State-Space Models - Analysis of time series (Linear Model Structures)

(CIV6540 - Probabilistic Machine Learning for Civil Engineers)

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Département des génies civil, géologique et des mines
Polytechnique Montréal

Chapter 12 – Goulet (2020)
Probabilistic Machine Learning for Civil Engineers
MIT Press

West, M. and Harrison, J. (1999)
Bayesian Forecasting and Dynamic Models
Springer

BDLM – General formulation

Transition model

$$\mathbf{x}_t = \mathbf{A}\mathbf{x}_{t-1} + \underbrace{\mathbf{w}_t}_{W \sim \mathcal{N}(\mathbf{0}, \mathbf{Q})}$$

Observation model

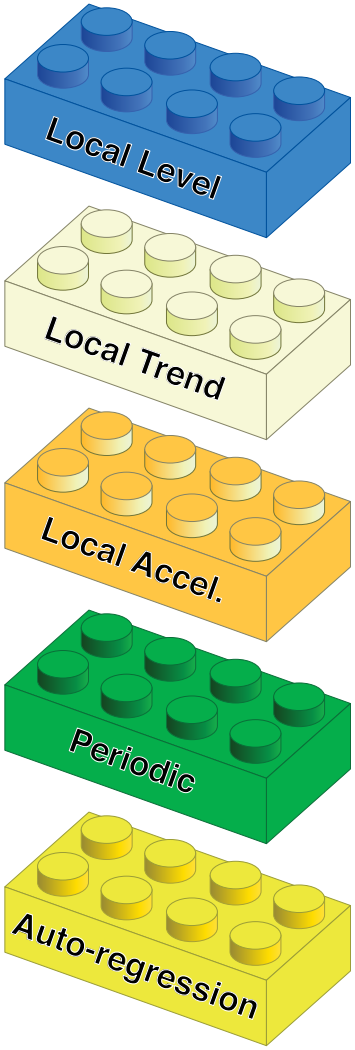
$$\mathbf{y}_t = \mathbf{C}\mathbf{x}_t + \underbrace{\mathbf{v}_t}_{V \sim \mathcal{N}(\mathbf{0}, \mathbf{R})}$$

Transition and observation models are completely defined by

- **A**: Model transition matrix
- **C**: Model observation matrix
- **Q**: Model errors covariance matrix
- **R**: Measurement errors covariance matrix

For each generic sub-component, define A, C & Q

Bayesian Dynamic Linear Model (BDLM)



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[Google images]

Module #7 Outline

Introduction
Sub-components
Dependencies
Case-Study SKF
Canari
OpenIPDM
Summary

Topics organization

Background
- 1 Revision probability & linear algebra
- 2 Probability distributions

Machine Learning Basics
- 0 Introduction
- 3 Bayesian Estimation $p(A|B) = \frac{p(B|A)p(A)}{p(B)}$
- 4 MCMC sampling & Newton

Supervised learning
- 5 Regression
- 6 Classification
- 7 LSTM networks for time series

Unsupervised learning
- 7 State-space model for time-series

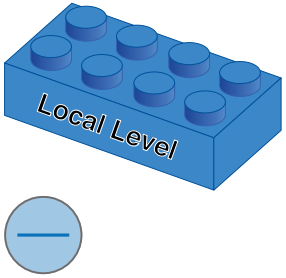
Decision Making & RL
- 8 Decision Theory
- 9 AI & Sequential decision problems

Section Outline

Sub-components

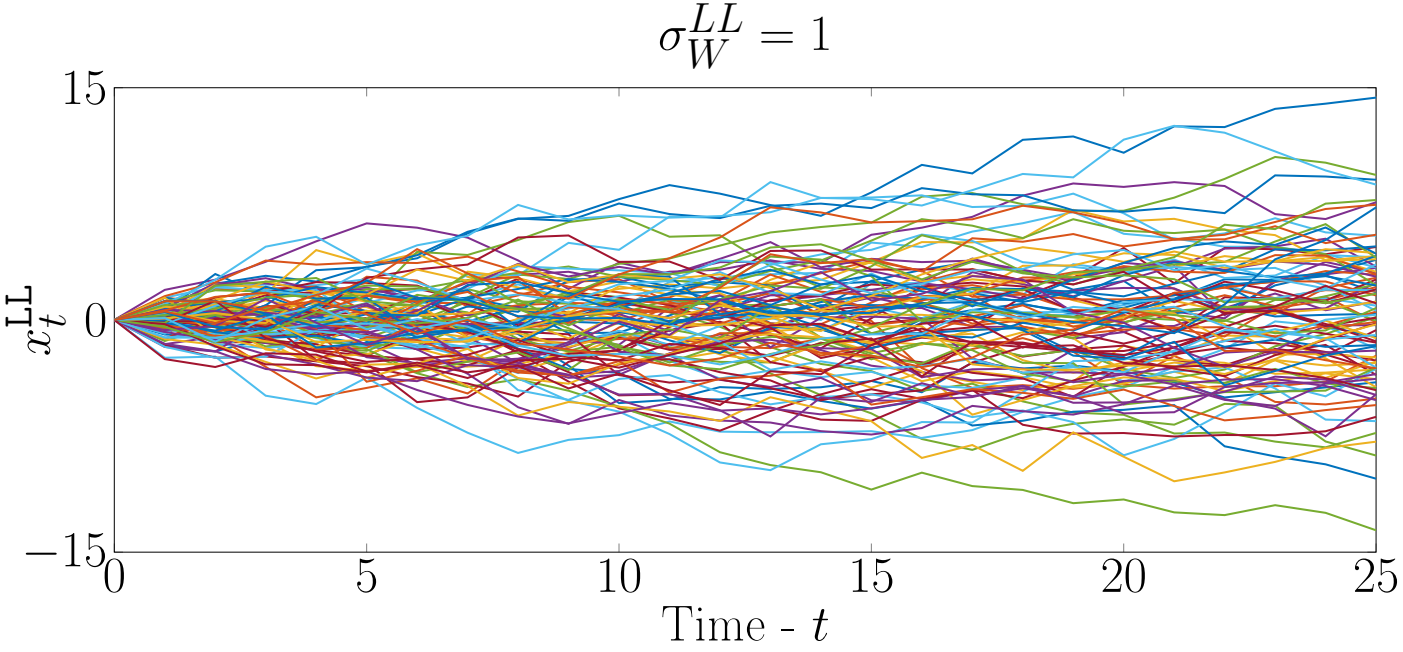
2.1 Generic sub-components
2.2 Assembling sub-components
2.3 Example - Monitoring Bridge Temperature

Local level [🐍]



$$\mathbf{x}^{\mathrm{LL}} = x^{\mathrm{LL}}$$

$$\mathbf{A}^{\mathrm{LL}} = 1,\ \mathbf{C}^{\mathrm{LL}} = 1,\ \mathbf{Q}^{\mathrm{LL}} = (\sigma_W^{\mathrm{LL}})^2$$



[CIV_ML/LL_demo.py]

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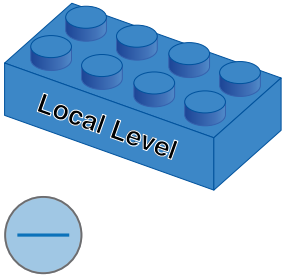

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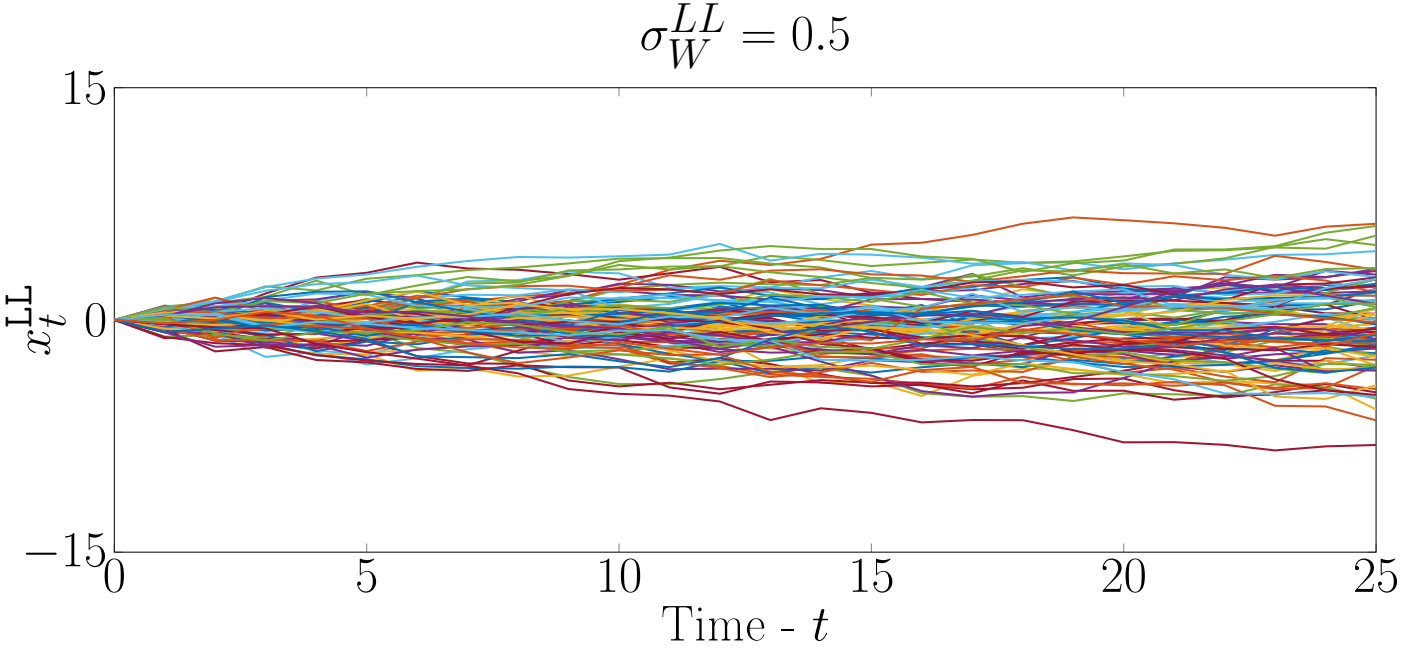

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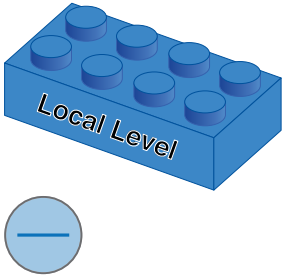

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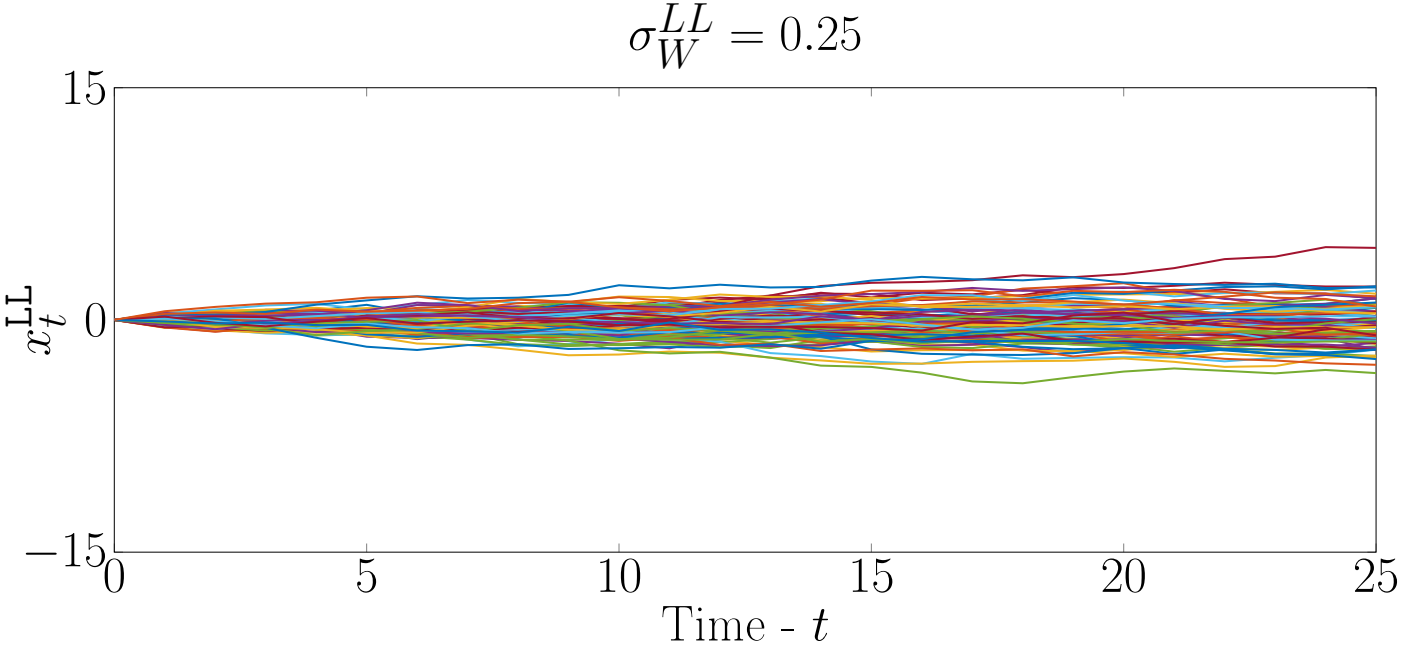

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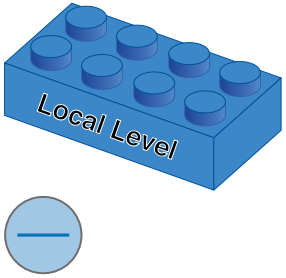

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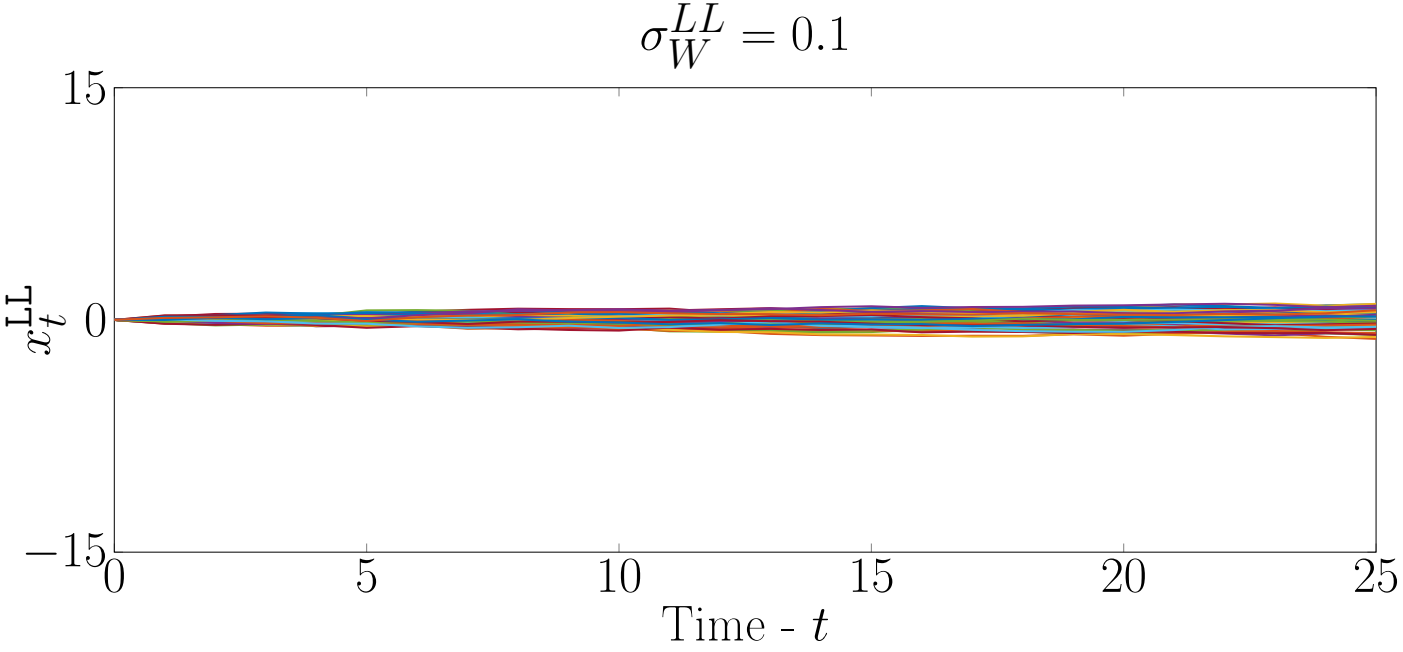

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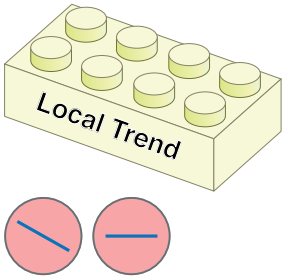
Local trend [🐍]



$$\mathbf{x}^{\mathrm{LT}} = \begin{bmatrix} x^{\mathrm{L}} \\ x^{\mathrm{LT}} \end{bmatrix}$$

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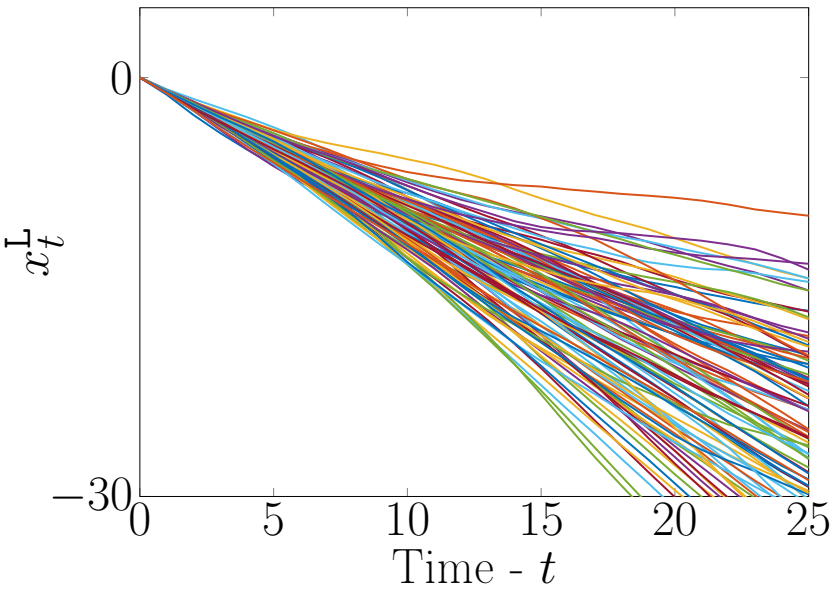

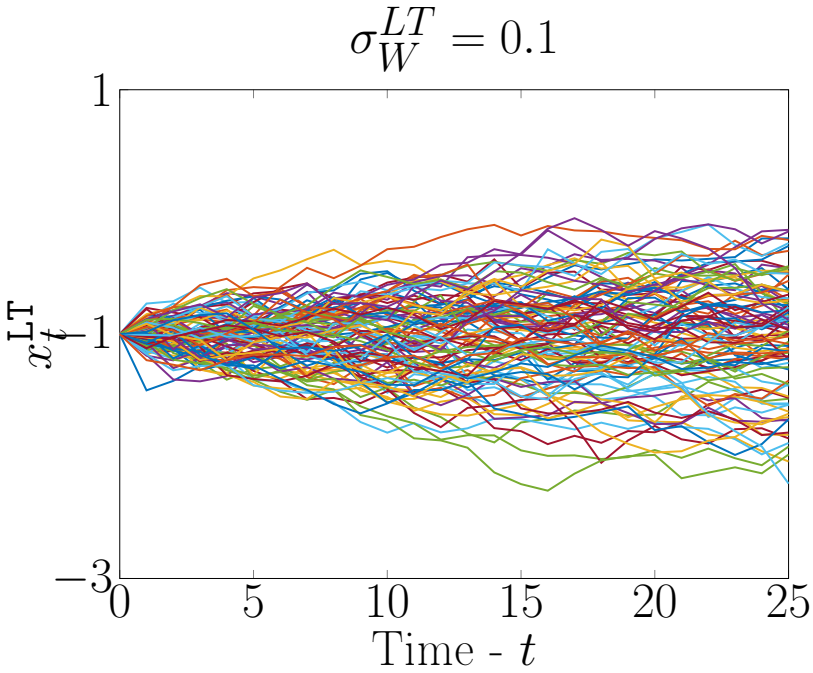

[CIV_ML/LT_demo.py]

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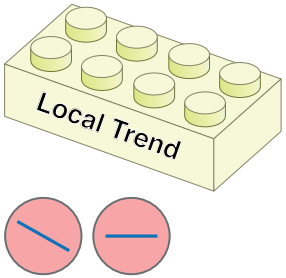

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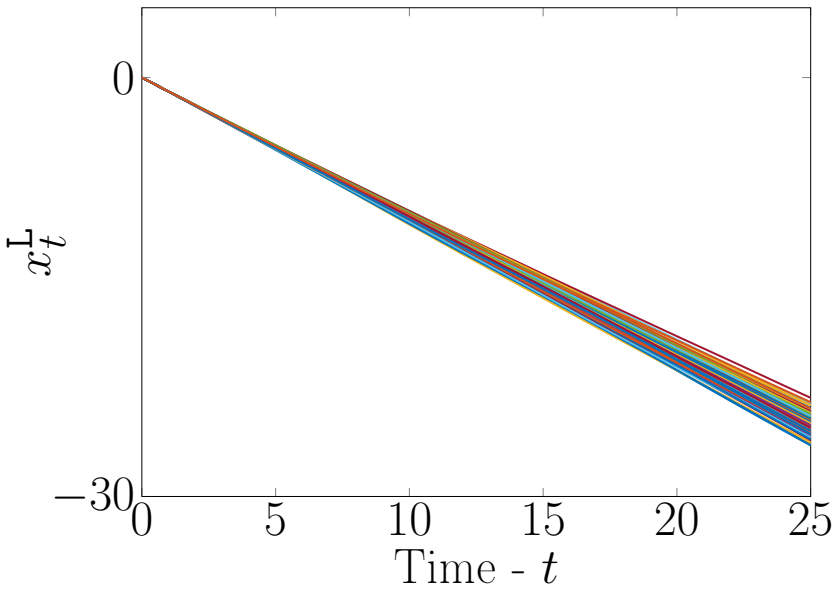

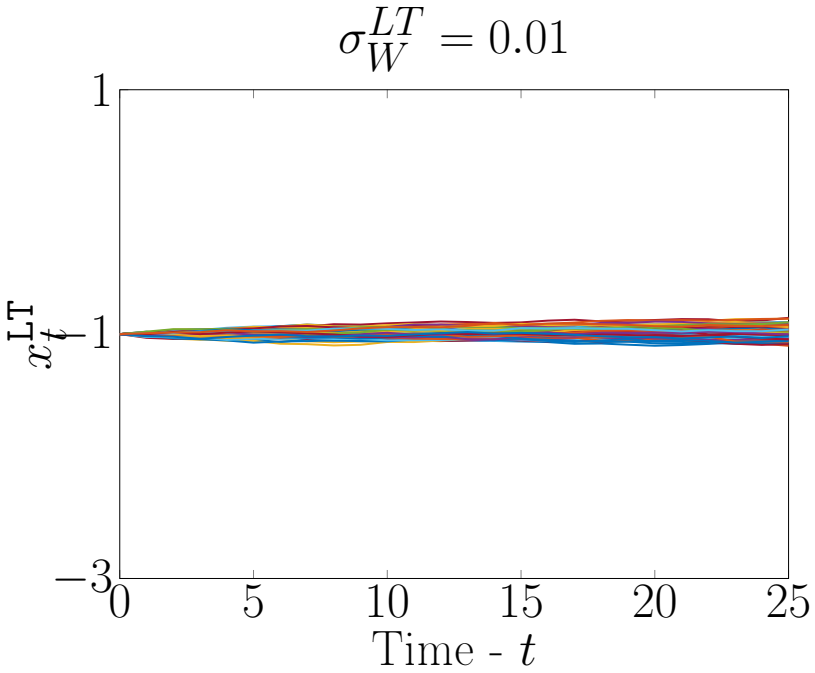

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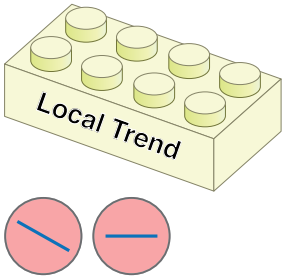

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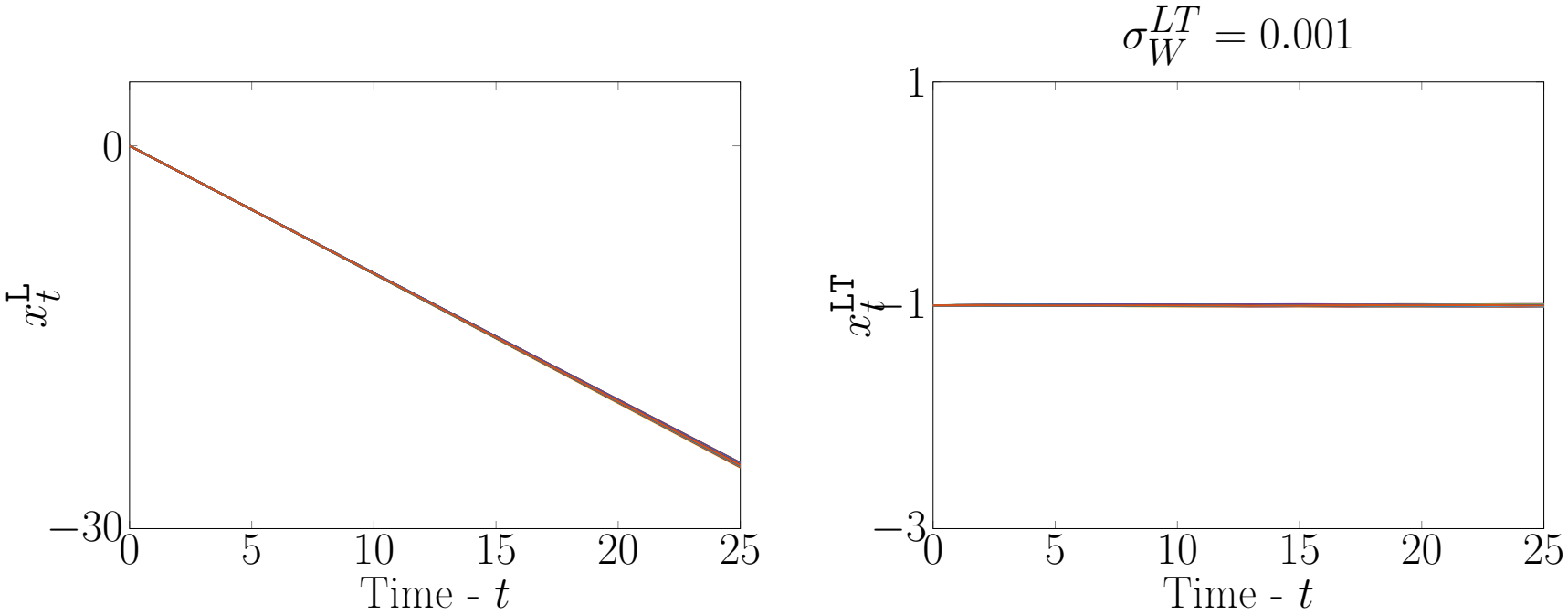


[CIV_ML/LT_demo.py]

Local trend & process error covariance matrix $\boldsymbol{Q}$

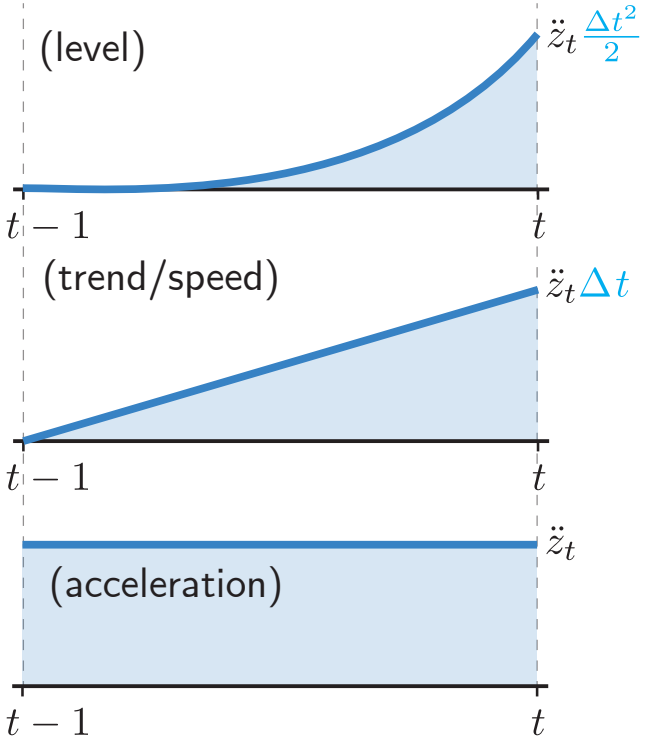
$$\mathbf{z}_t = [z_t \ \dot{z}_t \ \ddot{z}_t]^\mathsf{T} : \mathbf{Z}_t \sim \mathcal{N}(\mathbf{z}_t; \mathbf{0}, \boldsymbol{\Sigma}_\mathbf{Z})$$

$$\boldsymbol{\Sigma}_\mathbf{Z} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \sigma_W^2 \end{bmatrix}, \ \mathbf{A}_\mathbf{Z} = \begin{bmatrix} 0 & 0 & \frac{\Delta t^2}{2} \\ 0 & 0 & \Delta t \end{bmatrix}$$

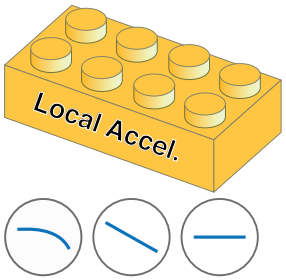
$$\mathbf{W}_t = \mathbf{A}_\mathbf{Z}\mathbf{Z}_t \sim \mathcal{N}(\mathbf{w}_t; \mathbf{0}, \underbrace{\mathbf{A}_\mathbf{Z}\boldsymbol{\Sigma}_\mathbf{Z}\mathbf{A}_\mathbf{Z}^\mathsf{T}}_{\mathbf{Q}}),$$

$$\mathbf{Q} = \mathbf{A}_\mathbf{Z}\boldsymbol{\Sigma}_\mathbf{Z}\mathbf{A}_\mathbf{Z}^\mathsf{T} = \sigma_W^2 \begin{bmatrix} \frac{\Delta t^4}{4} & \frac{\Delta t^3}{2} \\ \frac{\Delta t^3}{2} & \Delta t^2 \end{bmatrix}$$

$\boldsymbol{w}_t$ are caused by a **constant acceleration** $\ddot{z}_t$ over Δt



Local acceleration [🐍]



$$\mathbf{x}^{\mathtt{LA}} = \begin{bmatrix} x^{\mathtt{L}} \\ x^{\mathtt{T}} \\ x^{\mathtt{LA}} \end{bmatrix}$$

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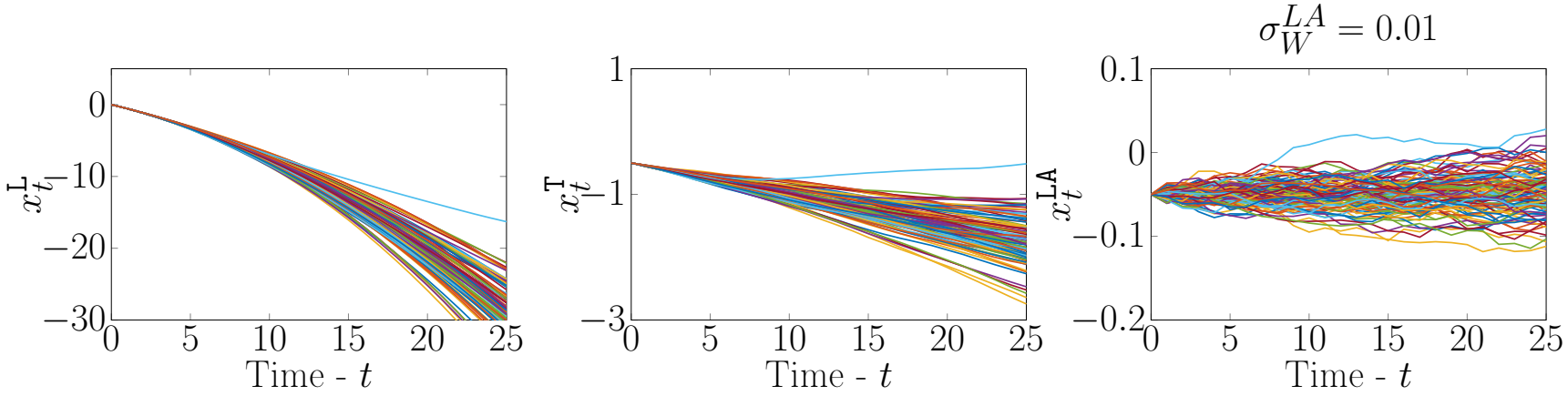

[CIV_ML/LA_demo.py]

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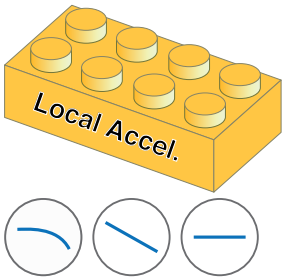

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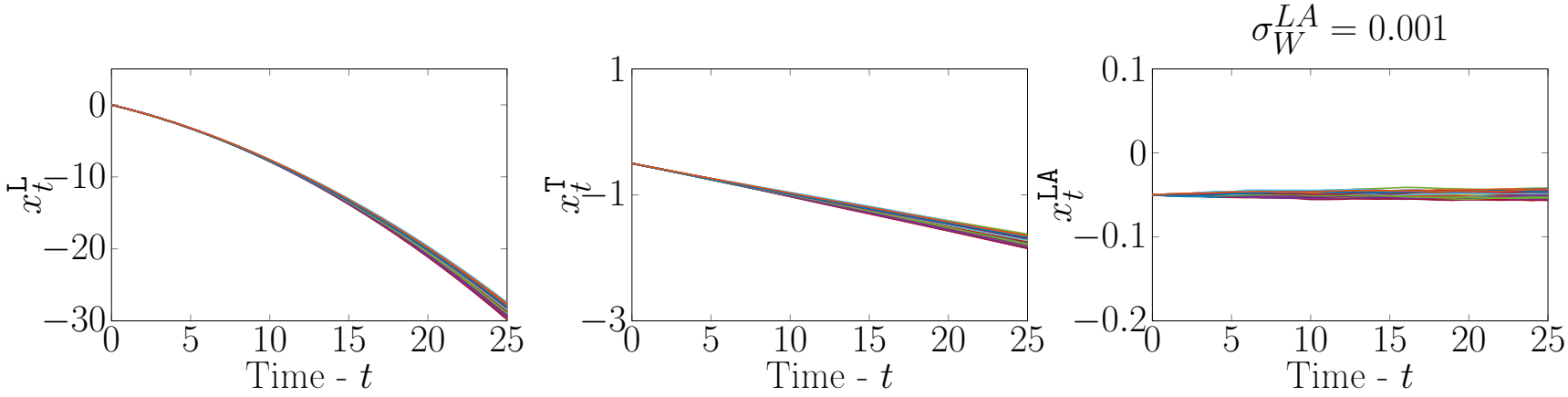

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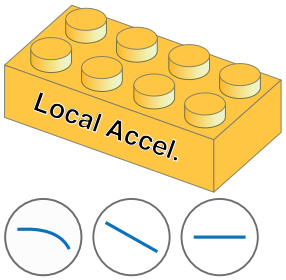

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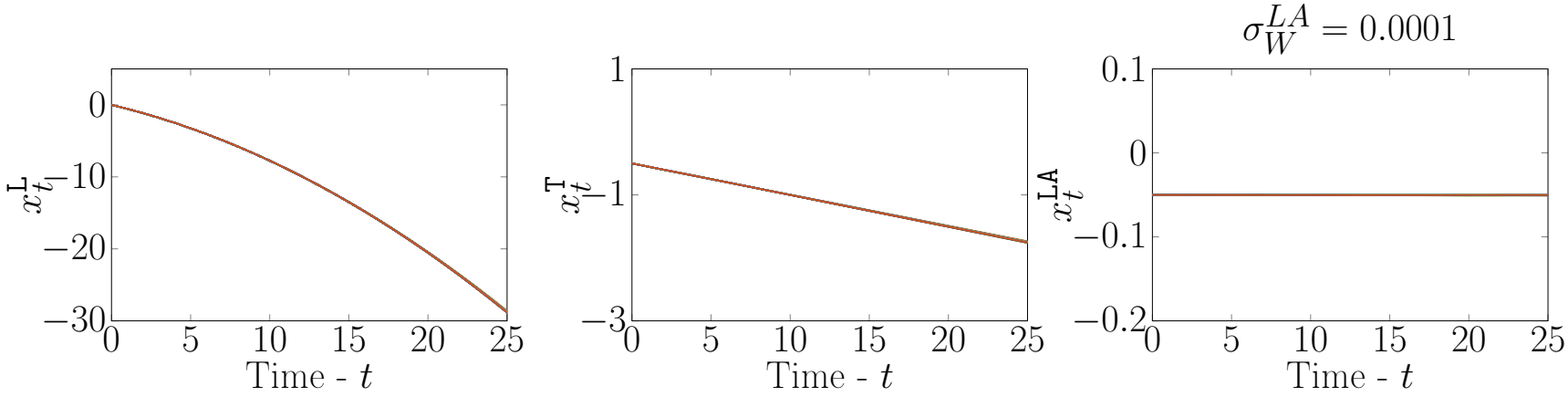
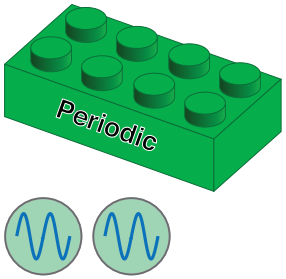


[CIV_ML/LA_demo.py]

Periodic [🐍]



Periodic components expressed in their **Fourrier form**

$$\mathbf{x}^{\mathrm{S}} = \left[\begin{array}{c} x^{\mathrm{S}_1} \\ x^{\mathrm{S}_2} \end{array}\right] \Bigg\} \; x^{\mathrm{S}_1}\text{:Amplitude}, \;\; \omega = \frac{2\pi\Delta t}{p}\text{: Frequency}$$

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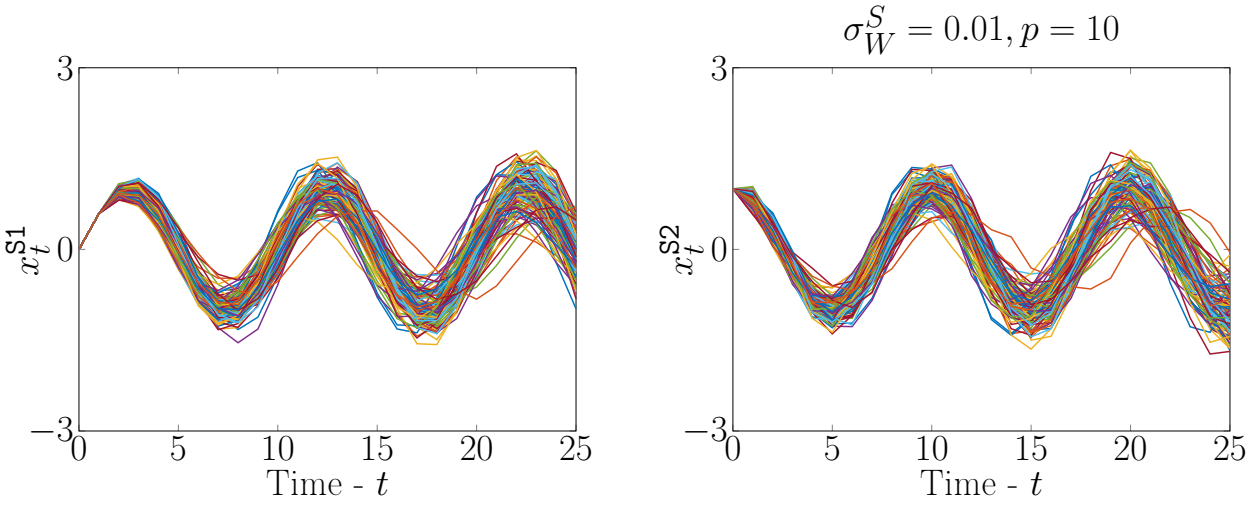

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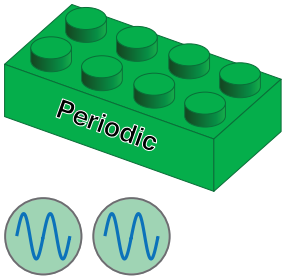

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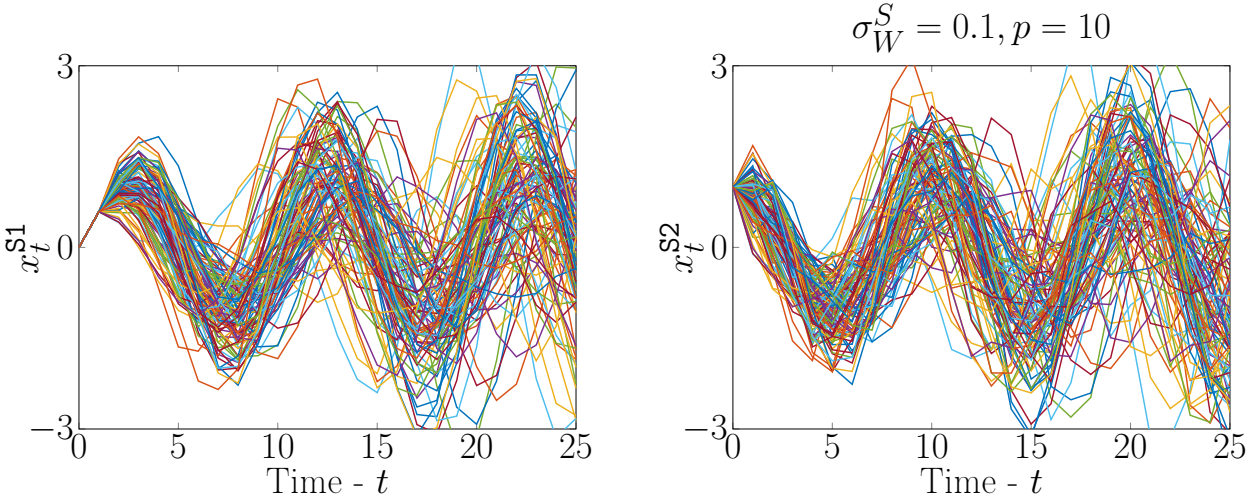

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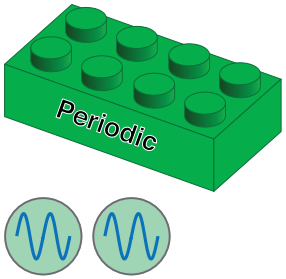

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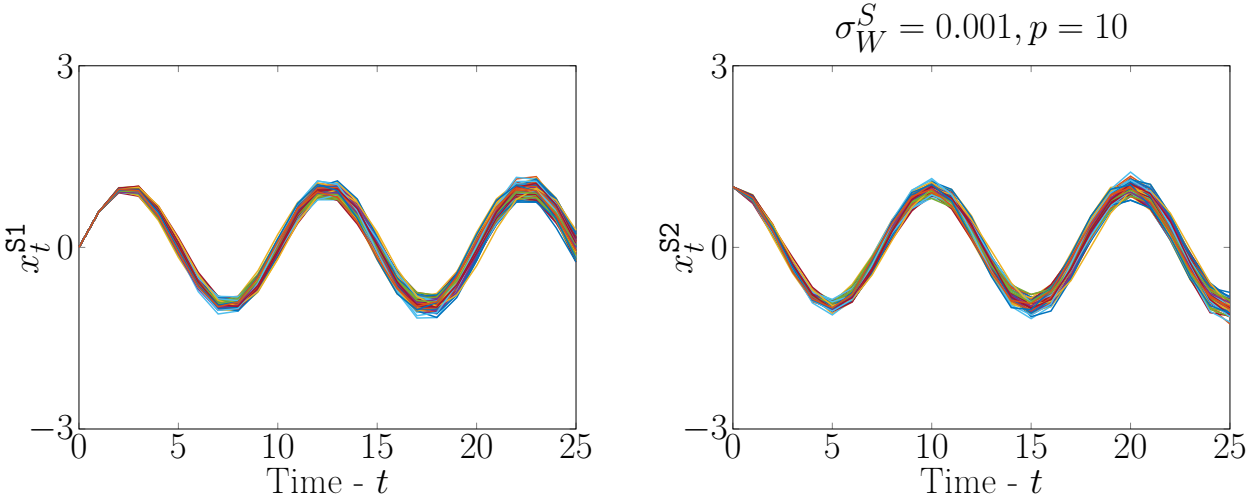

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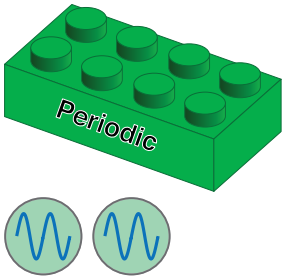

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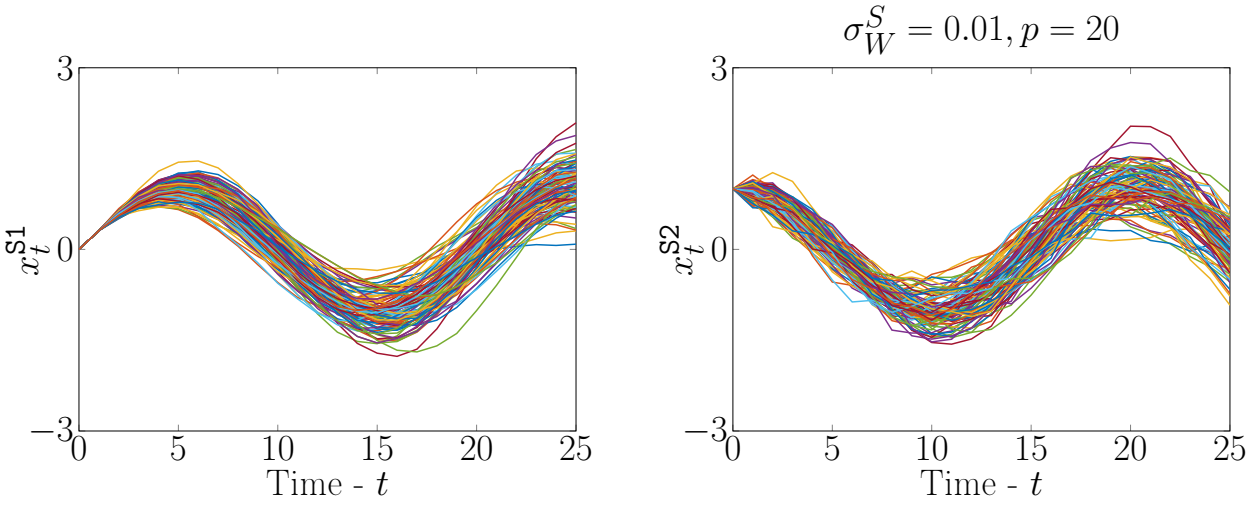
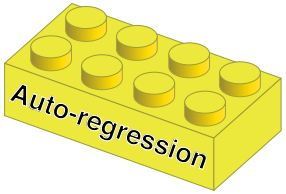


[CIV_ML/S_demo.py]

Autoregressive [🐍]

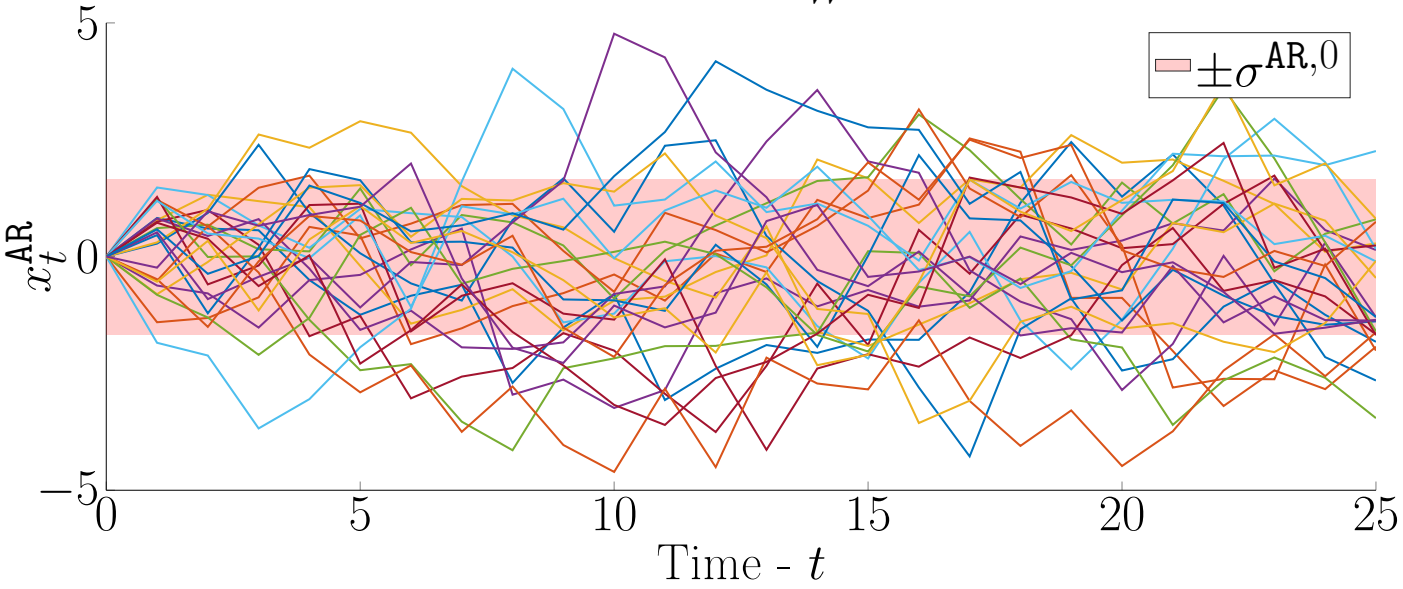


$$\mathbf{x}^{\mathtt{AR}} = x^{\mathtt{AR}}$$

$$\mathbf{A}^{\mathtt{AR}} = \phi^{\mathtt{AR}},\ \mathbf{C}^{\mathtt{AR}} = 1,\ \mathbf{Q}^{\mathtt{AR}} = (\sigma^{\mathtt{AR}})^2$$

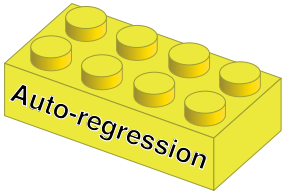
$0 < \phi^{\mathtt{AR}} < 1$: stationary process $\rightarrow (\sigma^{\mathtt{AR},0})^2 = \frac{(\sigma^{\mathtt{AR}})^2}{1-(\phi^{\mathtt{AR}})^2}$

$$\phi^{AR} = 0.8,\ \sigma_W^{AR} = 1$$



[CIV_ML/AR_demo.py]

Autoregressive [🐍]

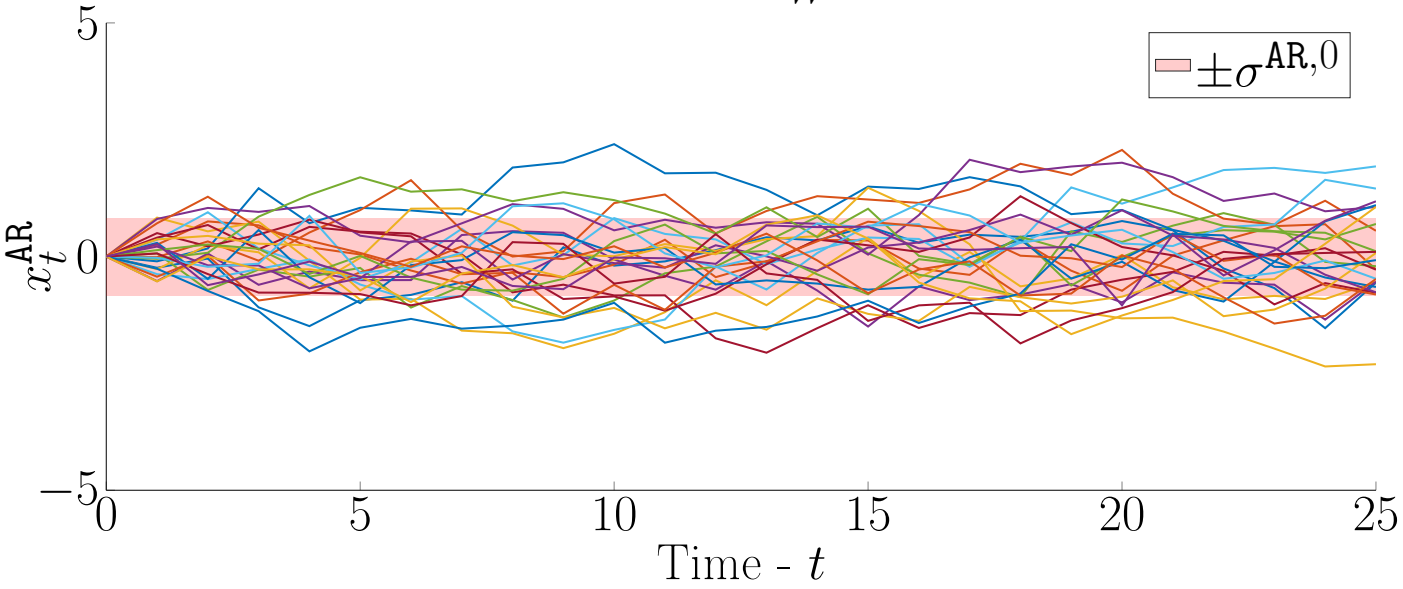


$$\mathbf{x}^{\mathtt{AR}} = x^{\mathtt{AR}}$$

$$\mathbf{A}^{\mathtt{AR}} = \phi^{\mathtt{AR}},\ \mathbf{C}^{\mathtt{AR}} = 1,\ \mathbf{Q}^{\mathtt{AR}} = (\sigma^{\mathtt{AR}})^2$$

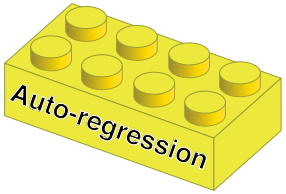
$0 < \phi^{\mathtt{AR}} < 1$: stationary process $\rightarrow (\sigma^{\mathtt{AR},0})^2 = \frac{(\sigma^{\mathtt{AR}})^2}{1-(\phi^{\mathtt{AR}})^2}$

$$\phi^{AR} = 0.8,\ \sigma_W^{AR} = 0.5$$



[CIV_ML/AR_demo.py]

Autoregressive [🐍]

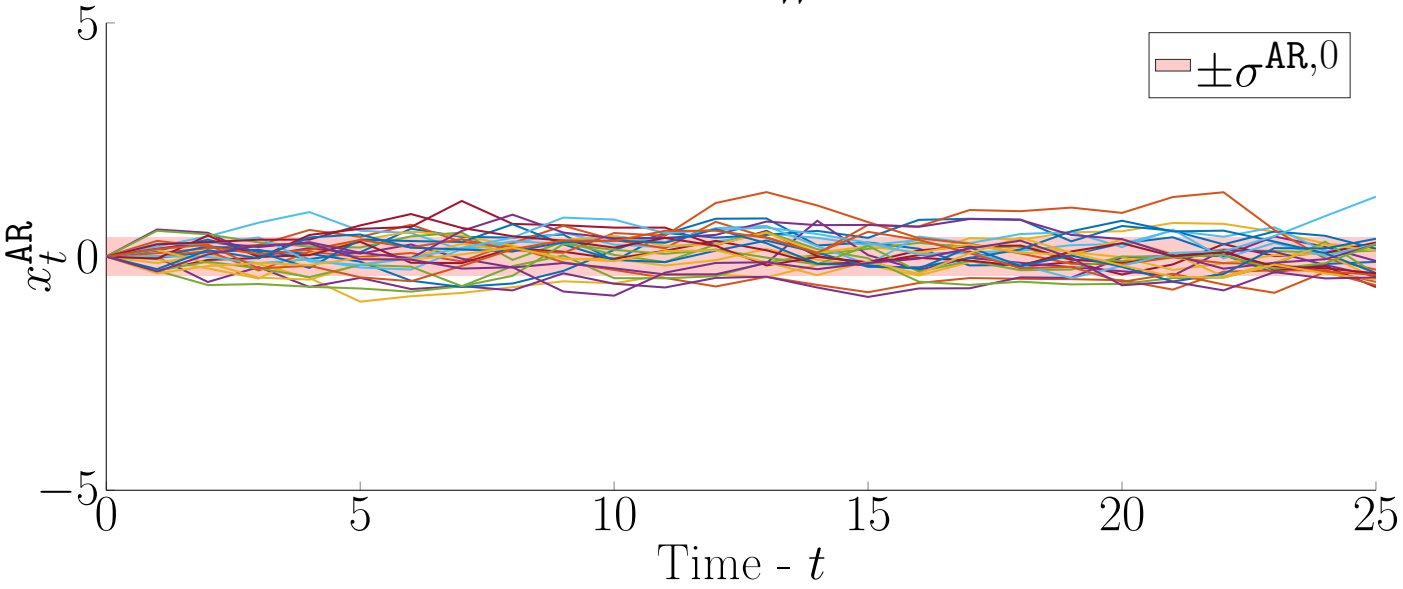


$$\mathbf{x}^{\mathtt{AR}} = x^{\mathtt{AR}}$$

$$\mathbf{A}^{\mathtt{AR}} = \phi^{\mathtt{AR}},\ \mathbf{C}^{\mathtt{AR}} = 1,\ \mathbf{Q}^{\mathtt{AR}} = (\sigma^{\mathtt{AR}})^2$$

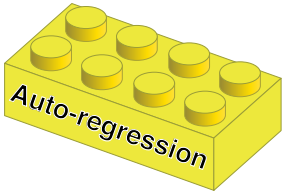
$0 < \phi^{\mathtt{AR}} < 1$: stationary process $\rightarrow (\sigma^{\mathtt{AR},0})^2 = \frac{(\sigma^{\mathtt{AR}})^2}{1-(\phi^{\mathtt{AR}})^2}$

$$\phi^{AR} = 0.8,\ \sigma_W^{AR} = 0.25$$



[CIV_ML/AR_demo.py]

Autoregressive [🐍]

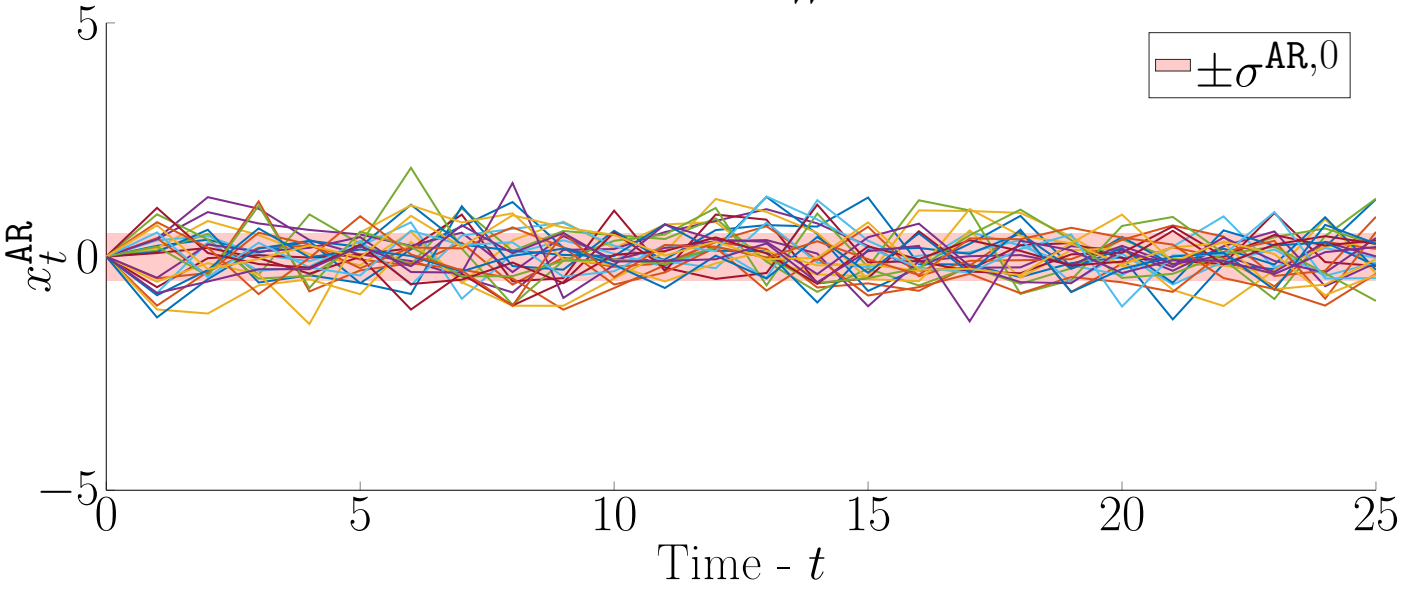


$$\mathbf{x}^{\mathtt{AR}} = x^{\mathtt{AR}}$$

$$\mathbf{A}^{\mathtt{AR}} = \phi^{\mathtt{AR}},\ \mathbf{C}^{\mathtt{AR}} = 1,\ \mathbf{Q}^{\mathtt{AR}} = (\sigma^{\mathtt{AR}})^2$$

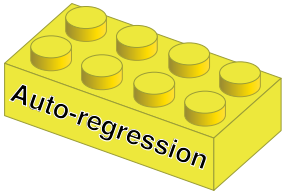
$0 < \phi^{\mathtt{AR}} < 1$: stationary process $\rightarrow (\sigma^{\mathtt{AR},0})^2 = \frac{(\sigma^{\mathtt{AR}})^2}{1-(\phi^{\mathtt{AR}})^2}$

$$\phi^{AR} = 0.2,\ \sigma_W^{AR} = 0.5$$



[CIV_ML/AR_demo.py]

Autoregressive [🐍]

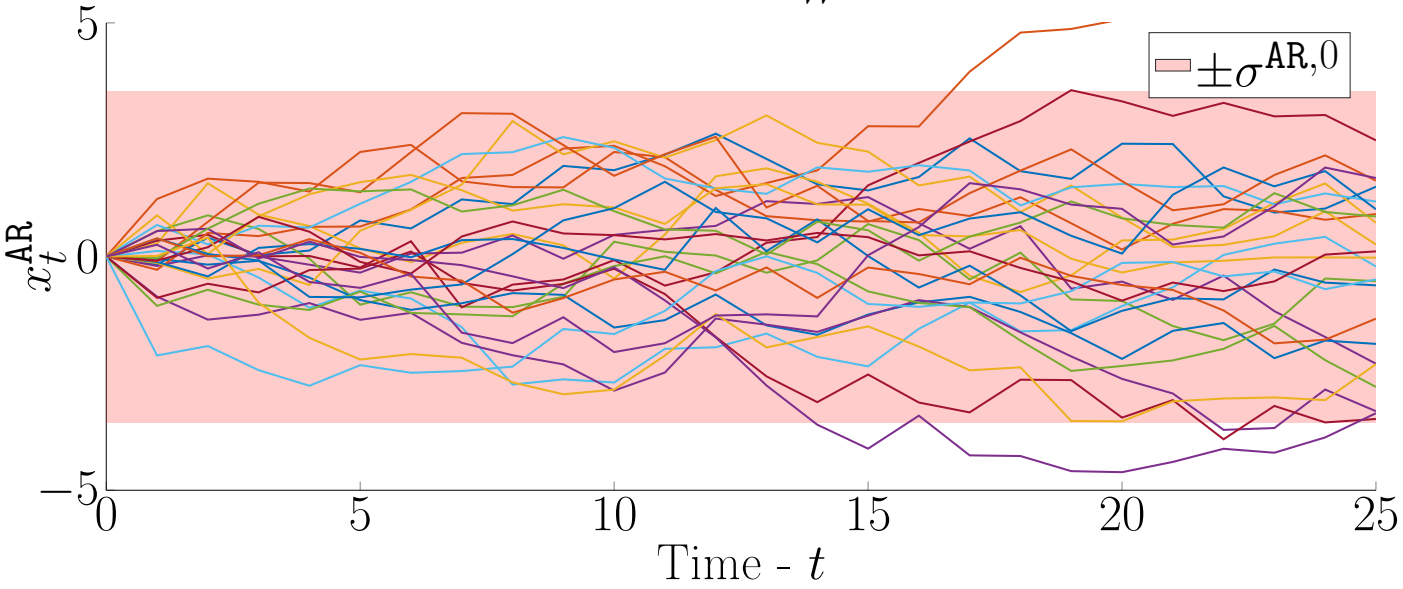


$$\mathbf{x}^{\mathtt{AR}} = x^{\mathtt{AR}}$$

$$\mathbf{A}^{\mathtt{AR}} = \phi^{\mathtt{AR}},\ \mathbf{C}^{\mathtt{AR}} = 1,\ \mathbf{Q}^{\mathtt{AR}} = (\sigma^{\mathtt{AR}})^2$$

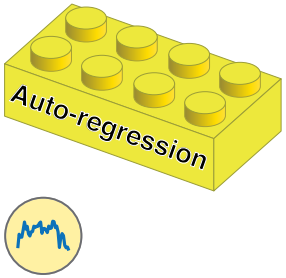
$0 < \phi^{\mathtt{AR}} < 1$: stationary process $\rightarrow (\sigma^{\mathtt{AR},0})^2 = \frac{(\sigma^{\mathtt{AR}})^2}{1-(\phi^{\mathtt{AR}})^2}$

$$\phi^{AR} = 0.99,\ \sigma_W^{AR} = 0.5$$



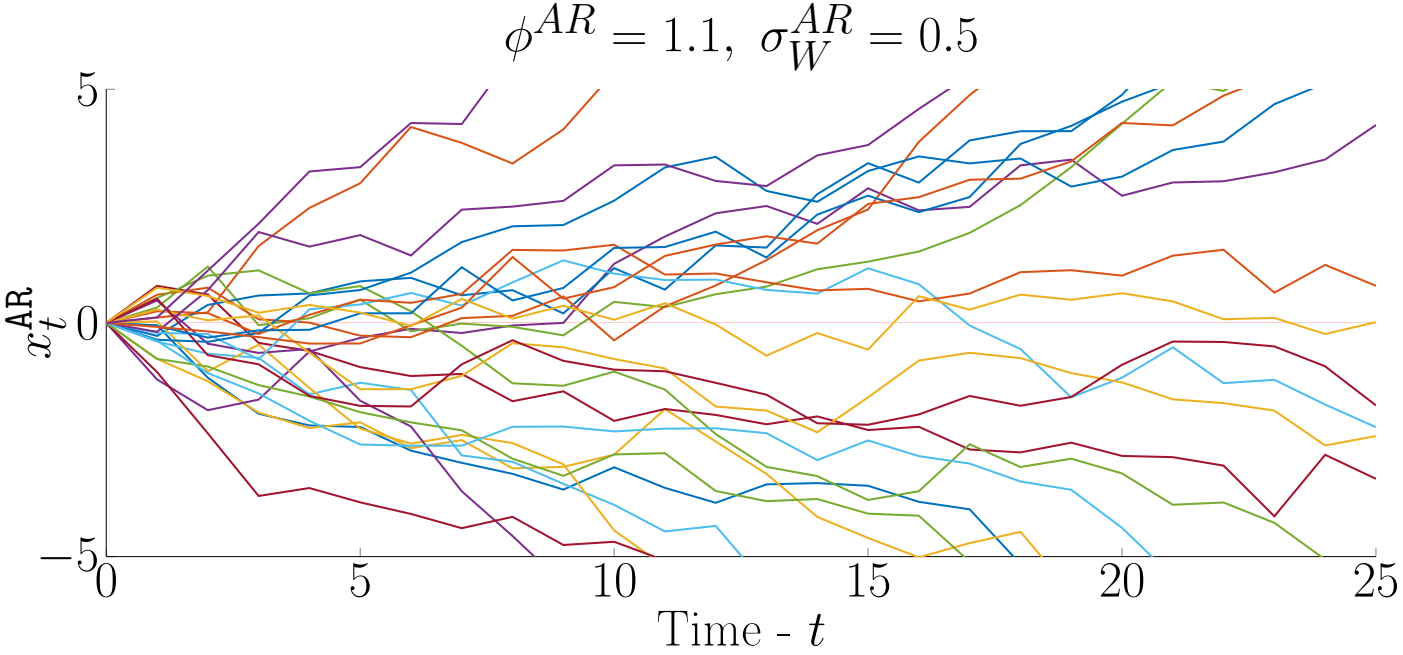
[CIV_ML/AR_demo.py]

Autoregressive [🐍]



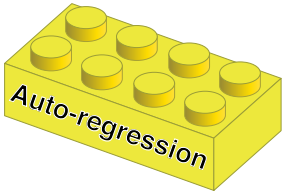
$$\mathbf{x}^{\mathtt{AR}} = x^{\mathtt{AR}}$$

$$\mathbf{A}^{\mathtt{AR}} = \phi^{\mathtt{AR}},\ \mathbf{C}^{\mathtt{AR}} = 1,\ \mathbf{Q}^{\mathtt{AR}} = (\sigma^{\mathtt{AR}})^2$$



[CIV_ML/AR_demo.py]

Autoregressive [🐍]

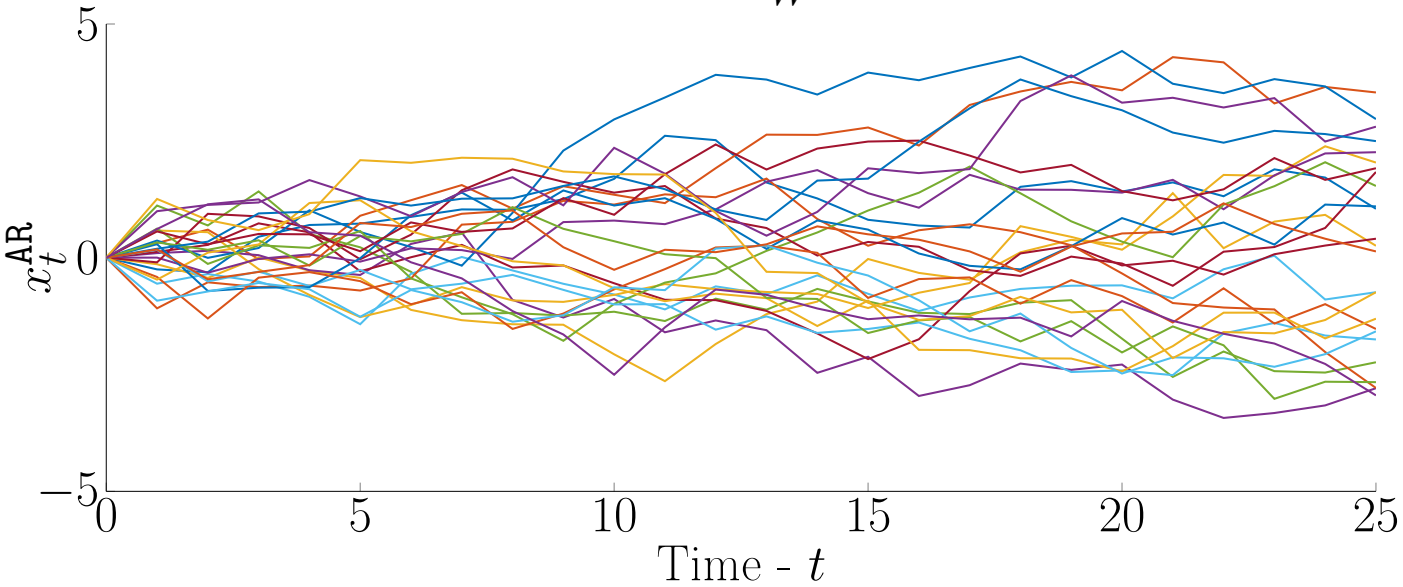


$$\mathbf{x}^{\mathtt{AR}} = x^{\mathtt{AR}}$$

$$\mathbf{A}^{\mathtt{AR}} = \phi^{\mathtt{AR}},\ \mathbf{C}^{\mathtt{AR}} = 1,\ \mathbf{Q}^{\mathtt{AR}} = (\sigma^{\mathtt{AR}})^2$$

$\phi^{\mathtt{AR}} \geq 1$: non-stationary process $\phi^{\mathtt{AR}} = 1$ corresponds to a random walk ($\equiv$ Local Level)

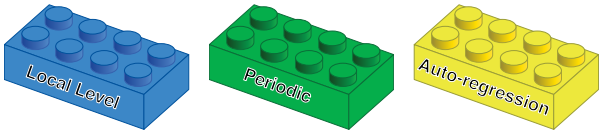
$$\phi^{AR} = 1,\ \sigma_W^{AR} = 0.5$$



[CIV_ML/AR_demo.py]

Assembling sub-components

For a set of sub-components:



Hidden state variables: $\mathbf{x} = [\underbrace{x^{\mathtt{LL}}}_{}, \underbrace{x^{\mathtt{S}_1}, x^{\mathtt{S}_2}}_{}, \underbrace{x^{\mathtt{AR}}}_{}]^{\mathsf{T}}$

Observation and Transition Model matrices:

$$\left.\begin{aligned}
\mathbf{C} &= [\mathbf{C}^{\mathtt{LL}}, \mathbf{C}^{\mathtt{S}}, \mathbf{C}^{\mathtt{AR}}] \\
\mathbf{A} &= \mathrm{blkdiag}(\mathbf{A}^{\mathtt{LL}}, \mathbf{A}^{\mathtt{S}}, \mathbf{A}^{\mathtt{AR}}) \\
\mathbf{Q} &= \mathrm{blkdiag}(\mathbf{Q}^{\mathtt{LL}}, \mathbf{Q}^{\mathtt{S}}, \mathbf{Q}^{\mathtt{AR}}) \\
\mathbf{R} &= \sigma_V^2
\end{aligned}\right\}
\begin{aligned}
\mathbf{x}_t &= \mathbf{A}\mathbf{x}_{t-1} + \underbrace{\mathbf{w}_t}_{W \sim \mathcal{N}(\mathbf{0}, \mathbf{Q})} \\
\mathbf{y}_t &= \mathbf{C}\mathbf{x}_t + \underbrace{\mathbf{v}_t}_{V \sim \mathcal{N}(\mathbf{0}, \mathbf{R})}
\end{aligned}$$

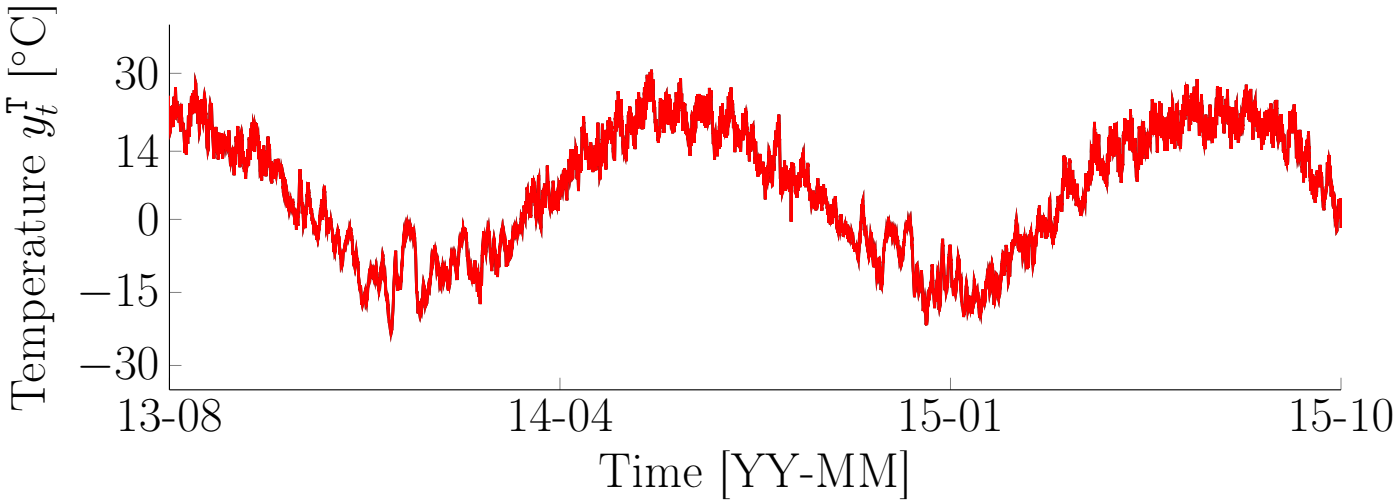
Block diagonal matrix (Linear Algebra [ ])

A block diagonal matrix concatenates several matrices on the diagonal of a global matrix.

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix}$$

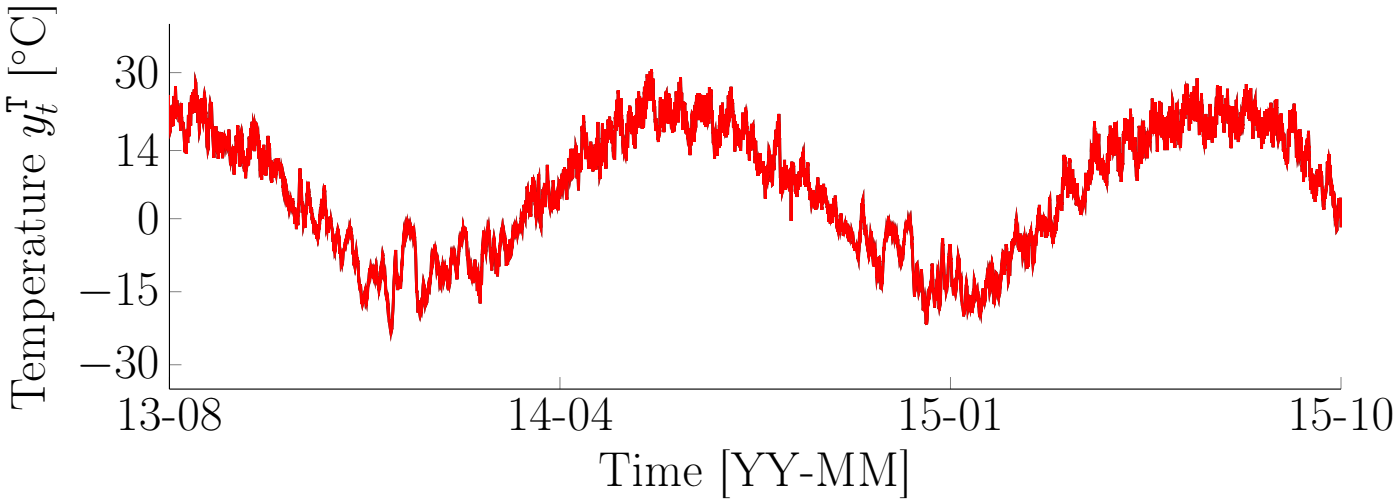
$$\text{blkdiag}(\mathbf{A}, \mathbf{B}) = \begin{bmatrix} 1 & 2 & 0 & 0 & 0 \\ 3 & 4 & 0 & 0 & 0 \\ 0 & 0 & 4 & 5 & 6 \\ 0 & 0 & 7 & 8 & 9 \\ 0 & 0 & 10 & 11 & 12 \end{bmatrix}$$

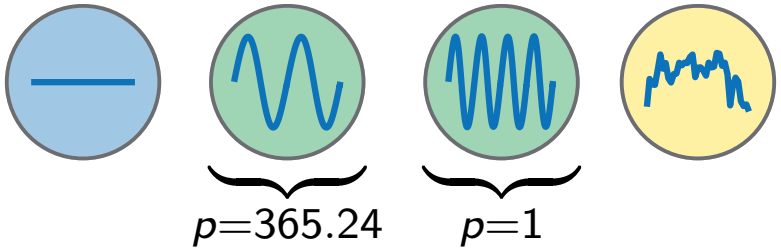
Temperature dataset



$$\mathbf{y}_{1:T} = [y_1, y_2, \cdots, y_T]$$

Hidden sub-components





Hidden state variables & model matrices

$$\mathbf{x}_t = [\underbrace{x_t^{\mathtt{LL}}}_{\text{—}}, \underbrace{x_t^{1,\mathtt{S}_1}, x_t^{1,\mathtt{S}_2}}_{\sim}, \underbrace{x_t^{2,\mathtt{S}_1}, x_t^{2,\mathtt{S}_2}}_{\sim\sim}, \underbrace{x_t^{\mathtt{AR}}}_{\sim\sim}]^\intercal$$

$$\mathbf{A} = \text{block diag}\left(1, \begin{bmatrix} \cos\omega^{\mathtt{T1}} & \sin\omega^{\mathtt{T1}} \\ -\sin\omega^{\mathtt{T1}} & \cos\omega^{\mathtt{T1}} \end{bmatrix}, \begin{bmatrix} \cos\omega^{\mathtt{T2}} & \sin\omega^{\mathtt{T2}} \\ -\sin\omega^{\mathtt{T2}} & \cos\omega^{\mathtt{T2}} \end{bmatrix}, \phi^{\mathtt{AR}}\right)$$

$$\mathbf{Q} = \text{block diag}\left([0, 0, 0, 0, 0, \sigma^{\mathtt{AR}}]^2\right)$$

$$\mathbf{C} = [1, 1, 0, 1, 0, 1]$$

$$\mathbf{R} = (\sigma^{\mathtt{V}})^2$$

$$\boldsymbol{\theta} = [\sigma^{\mathtt{AR}}, \sigma^{\mathtt{V}}, \phi^{\mathtt{AR}}]^\intercal \quad \text{(Unknown parameters)}$$

Initial hidden state mean and covariance

$$\mathbf{x}_t = [\underbrace{x_t^{\mathtt{LL}}}_{\text{—}}, \underbrace{x_t^{1,\mathtt{S}_1}, x_t^{1,\mathtt{S}_2}}_{\sim}, \underbrace{x_t^{2,\mathtt{S}_1}, x_t^{2,\mathtt{S}_2}}_{\sim\sim}, \underbrace{x_t^{\mathtt{AR}}}_{\sim\sim}]^\mathsf{T}$$

$$\mathbf{X}_0 \sim \mathcal{N}(\mathbf{x}_0; \boldsymbol{\mu}_0, \boldsymbol{\Sigma}_0) \left\{ \begin{array}{lcl} \boldsymbol{\mu}_0 & = & [6, 0, 0, 0, 0, 0]^\mathsf{T} \\ \boldsymbol{\Sigma}_0 & = & \mathrm{diag}([5, 10, 10, 10, 10, 1]^2) \end{array} \right.$$

Hidden state estimation

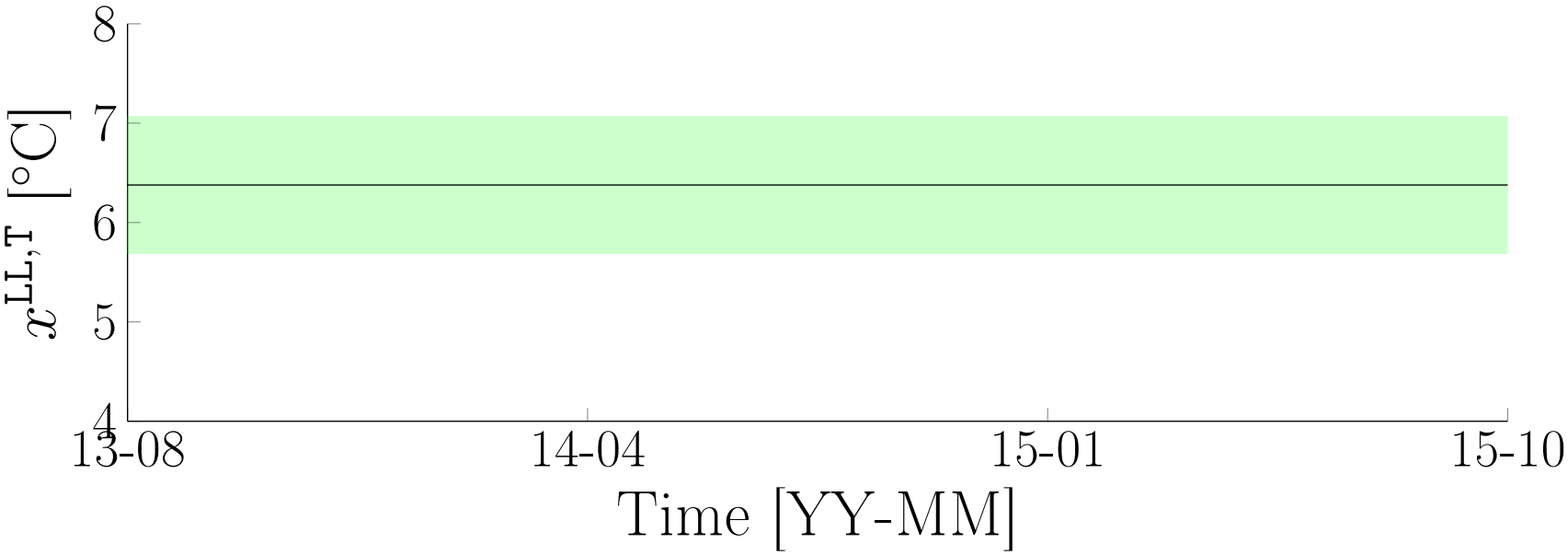
$$\underbrace{\left.\begin{matrix}\mathbf{A}\\ \mathbf{C}\\ \mathbf{Q}\\ \mathbf{R}\end{matrix}\right\}\begin{aligned}\mathbf{x}_t &= \mathbf{A}\mathbf{x}_{t-1} + \underbrace{\mathbf{w}_t}_{W\sim\mathcal{N}(\mathbf{0},\mathbf{Q})}\\ y_t &= \mathbf{C}\mathbf{x}_t + \underbrace{\mathbf{v}_t}_{V\sim\mathcal{N}(\mathbf{0},\mathbf{R})}\end{aligned}}_{\text{Kalman Filter}}$$

$$(\boldsymbol{\mu}_{t|t}, \boldsymbol{\Sigma}_{t|t}, \mathcal{L}_t) = \text{Filter}(\boldsymbol{\mu}_{t-1|t-1}, \boldsymbol{\Sigma}_{t-1|t-1}, y_t; \mathbf{A}, \mathbf{C}, \mathbf{Q}, \mathbf{R})$$

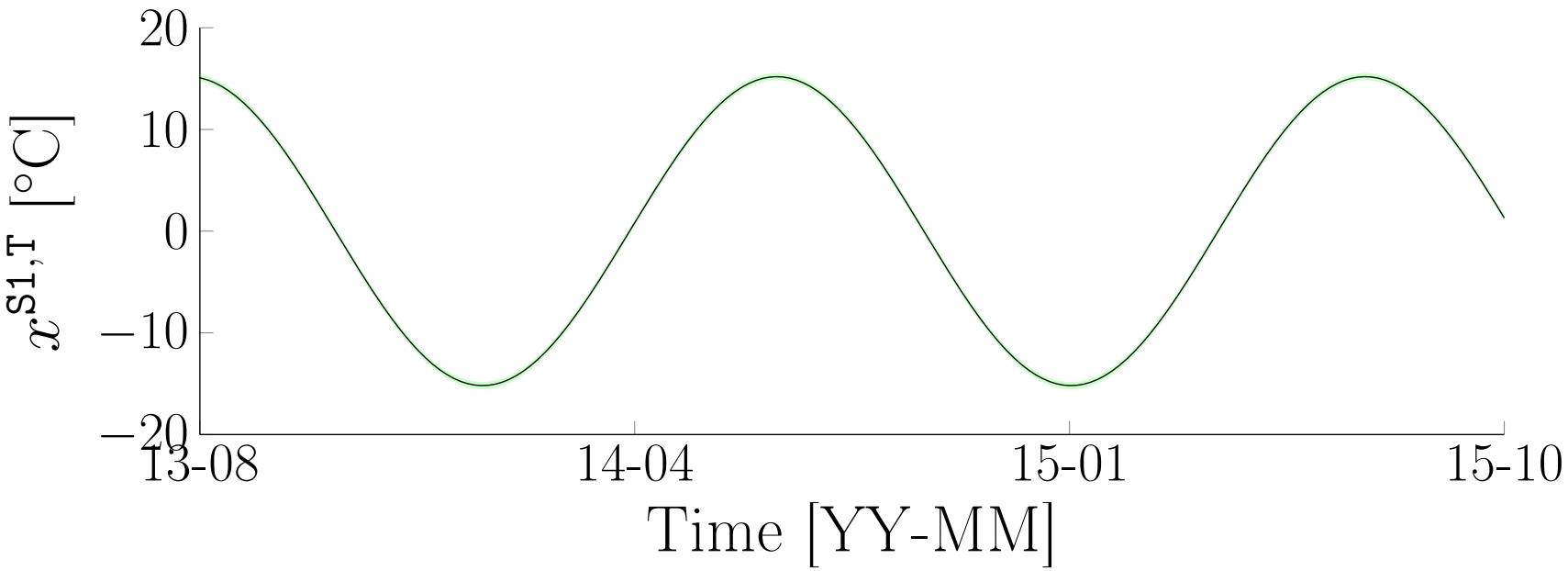
Note: Optimal parameter $\boldsymbol{\theta}^*$ must be identified using a maximization algorithm (e.g. Newton-Raphson)

$$\boldsymbol{\theta}^* = \arg\max_{\boldsymbol{\theta}} \sum_{t=1}^{T} \ln \mathcal{L}_t$$

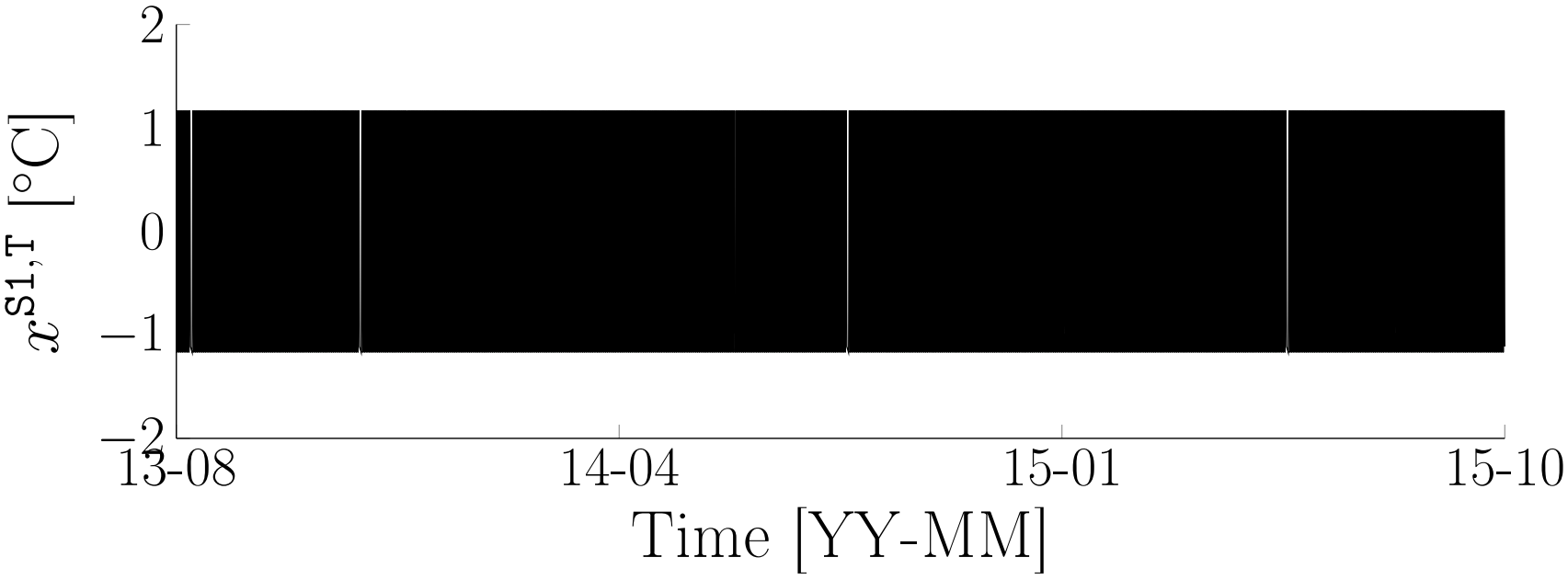
Hidden state estimation (cont.)



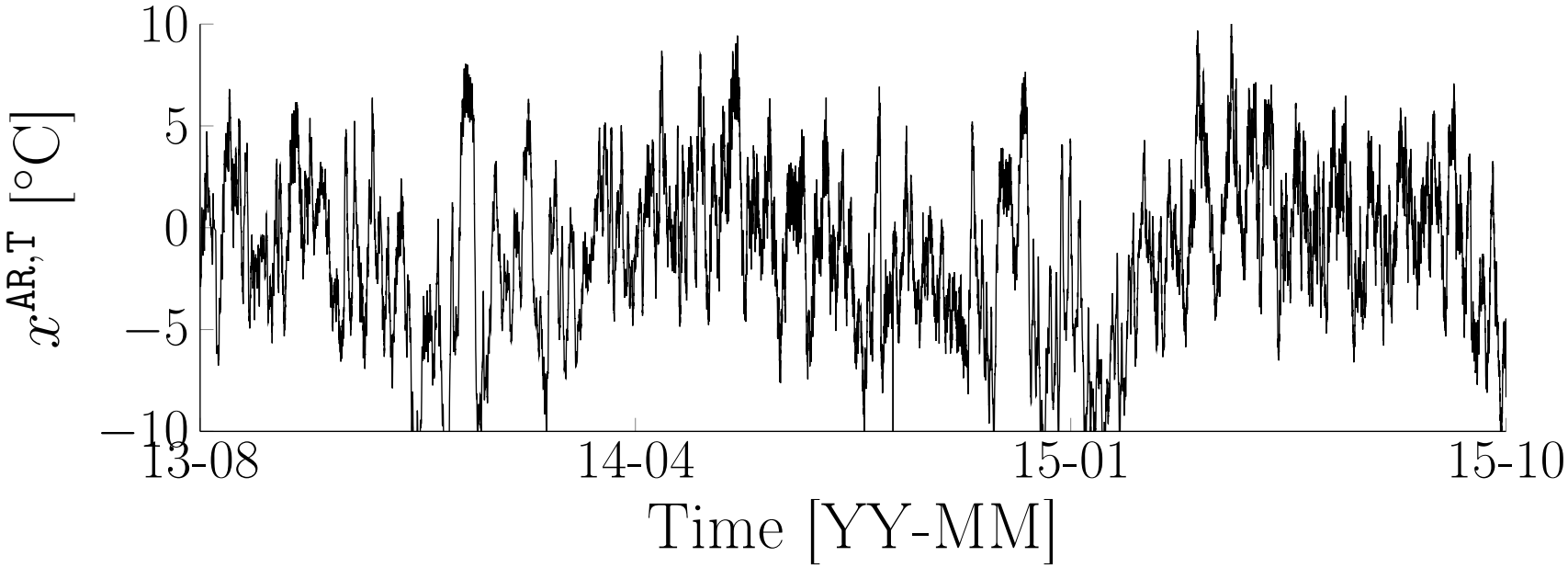
Hidden state estimation (cont.)



Hidden state estimation (cont.)



Hidden state estimation (cont.)



Section Outline

Dependencies

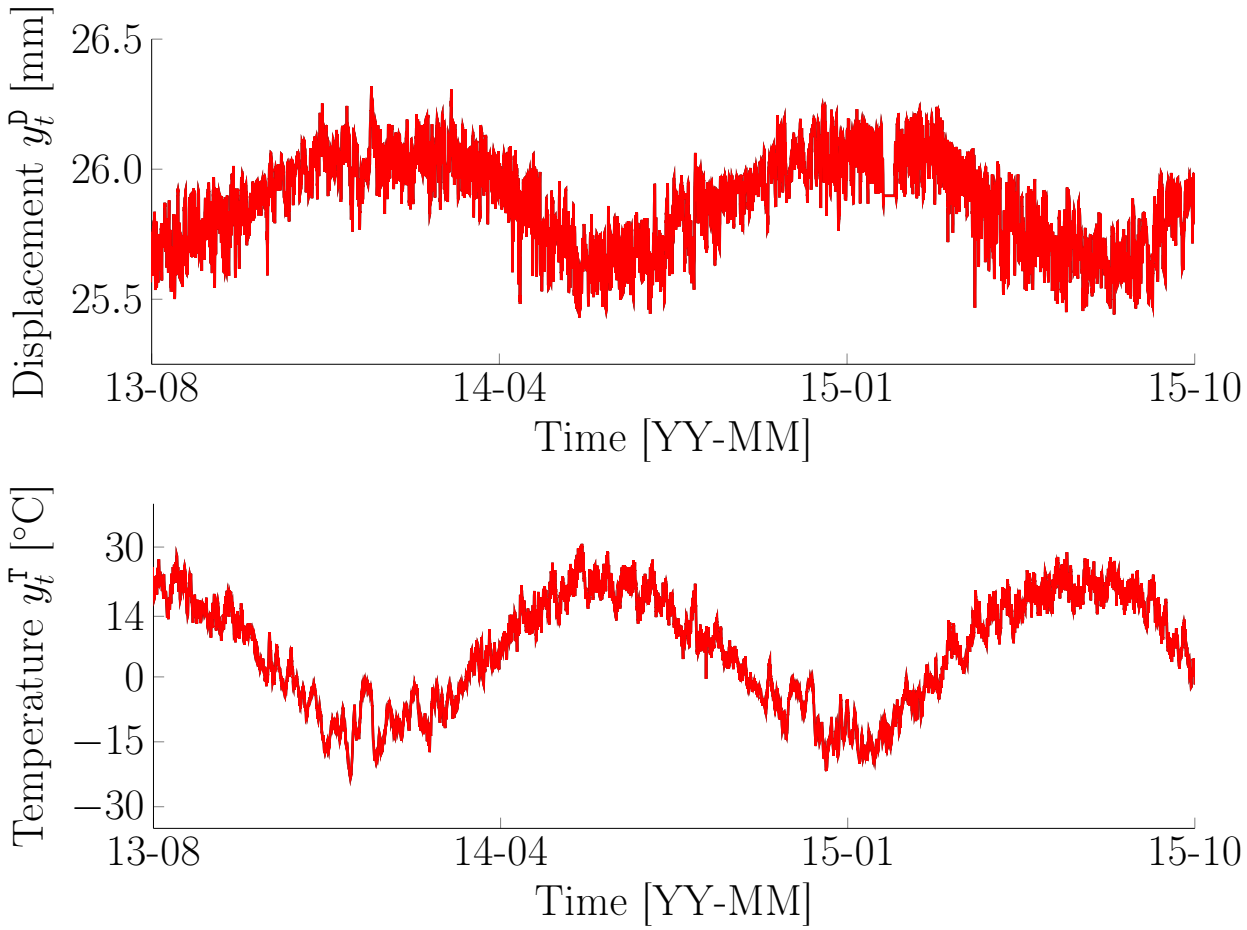
3.1 Context

3.2 Sub-components

3.3 Assembling sub-components

3.4 Exemple – Monitoring temperature & displacement

Example - Monitoring temperature & displacement

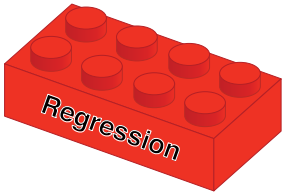


$$\mathbf{y}_{1:T} = \begin{bmatrix} \mathbf{y}_{1:T}^{\mathtt{D}} \\ \mathbf{y}_{1:T}^{\mathtt{T}} \end{bmatrix}$$

$\mathtt{D}$: Displacement

$\mathtt{T}$: Temperature

Regression



Regression components: **no state variable of its own.**

It expresses the dependence between any two sub-component by adding a term in the **observation matrix**,

$$[\mathbf{C}]_{ij} = \phi^{\mathtt{i}|\mathtt{j},\mathtt{R}}$$

where $\phi^{\mathtt{i}|\mathtt{j},\mathtt{R}} \in \mathbb{R}$ is a regression coefficient.

Assembling sub-components

$$
\begin{aligned}
\mathbf{C} &= \text{blkdiag}(\mathbf{C}^1, \mathbf{C}^2, \cdots, \mathbf{C}^{\mathtt{D}}) \\
\mathbf{A} &= \text{blkdiag}(\mathbf{A}^1, \mathbf{A}^2, \cdots, \mathbf{A}^{\mathtt{D}}) \\
\mathbf{Q} &= \text{blkdiag}(\mathbf{Q}^1, \mathbf{Q}^2, \cdots, \mathbf{Q}^{\mathtt{D}}) \\
\mathbf{R} &= \text{blkdiag}(\mathbf{R}^1, \mathbf{R}^2, \cdots, \mathbf{R}^{\mathtt{D}}).
\end{aligned}
$$

Assembling sub-components

$$
\begin{aligned}
\mathbf{C} &= \text{blkdiag}(\mathbf{C}^{\mathrm{D}}, \mathbf{C}^{\mathrm{T}}) \\
\mathbf{A} &= \text{blkdiag}(\mathbf{A}^{\mathrm{D}}, \mathbf{A}^{\mathrm{T}}) \\
\mathbf{Q} &= \text{blkdiag}(\mathbf{Q}^{\mathrm{D}}, \mathbf{Q}^{\mathrm{T}}) \\
\mathbf{R} &= \text{blkdiag}(\mathbf{R}^{\mathrm{D}}, \mathbf{R}^{\mathrm{T}}).
\end{aligned}
$$

Assembling sub-components

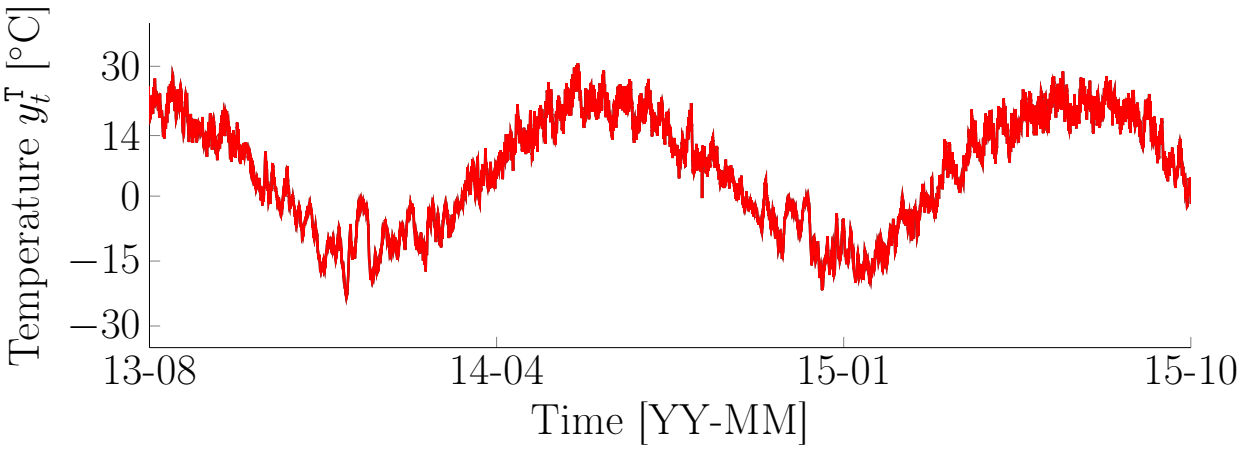
$$\mathbf{C} = \overbrace{\begin{bmatrix} \mathbf{C}^1 & \mathbf{C}^c_{1,2} & \cdots & \mathbf{C}^c_{1,j} & \cdots & \mathbf{C}^c_{1,\mathtt{D}} \\ \mathbf{C}^c_{2,1} & \mathbf{C}^2 & \cdots & \mathbf{C}^c_{2,j} & \cdots & \mathbf{C}^c_{2,\mathtt{D}} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{C}^c_{i,1} & \mathbf{C}^c_{i,2} & \cdots & \mathbf{C}^c_{i,j} & \cdots & \mathbf{C}^c_{i,\mathtt{D}} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{C}^c_{\mathtt{D},1} & \mathbf{C}^c_{\mathtt{D},2} & \cdots & \mathbf{C}^c_{\mathtt{D},j} & \cdots & \mathbf{C}^{\mathtt{D}} \end{bmatrix}}^{\text{Global observation matrix}} \qquad \mathbf{D} = \overbrace{\begin{bmatrix} 1 & d_{1,2} & \cdots & d_{1,j} & \cdots & d_{1,\mathtt{D}} \\ d_{2,1} & 1 & \cdots & d_{2,j} & \cdots & d_{2,\mathtt{D}} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ d_{i,1} & d_{i,2} & \cdots & 1 & \cdots & d_{i,\mathtt{D}} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ d_{\mathtt{D},1} & d_{\mathtt{D},2} & \cdots & d_{\mathtt{D},j} & \cdots & 1 \end{bmatrix}}^{\text{Dependence matrix}}$$

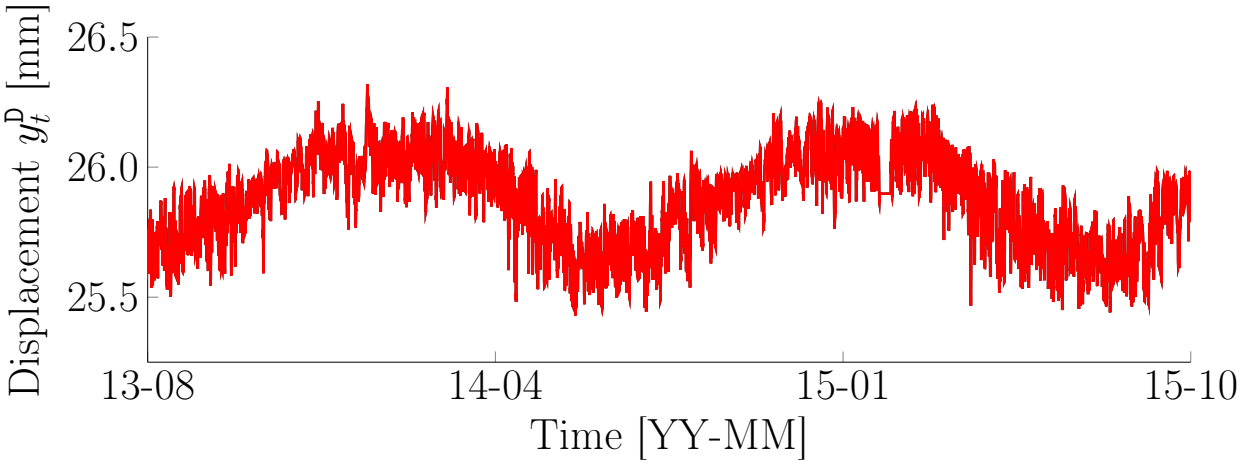
$$d_{i,j} = 0 \quad \rightarrow \quad \mathbf{C}^c_{i,j} = [\mathbf{0}]$$

$$d_{i,j} = 1 \quad \rightarrow \quad \mathbf{C}^c_{i,j} = \left[\phi_1^{i|j}, \phi_2^{i|j}, \cdots, \phi_{k_j}^{i|j}\right] \in \mathbb{R}^{k_j}$$

$\phi_k^{i|j}$ **regression coefficient** $\rightarrow$ linear dependence between the kth sub-component from the jth time series and the ith time series.

Example – Monitoring temperature & displacement



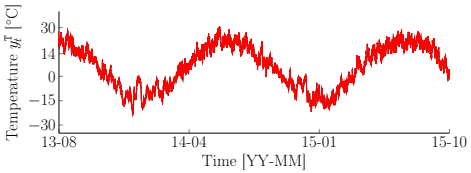


$$\mathbf{y}_{1:T} = \begin{bmatrix} \mathbf{y}_{1:T}^{\mathtt{D}} \\ \mathbf{y}_{1:T}^{\mathtt{T}} \end{bmatrix}$$

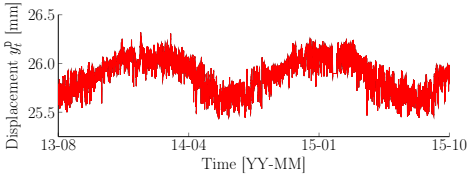
$\mathtt{D}$: Displacement

$\mathtt{T}$: Temperature

Hidden state variables



$$\mathbf{x}_t^{\mathtt{T}} = [\underbrace{x_t^{\mathtt{T,LL}}}_{}, \underbrace{x_t^{\mathtt{T1,S_1}}, x_t^{\mathtt{T1,S_2}}}_{p=365.24}, \underbrace{x_t^{\mathtt{T2,S_1}}, x_t^{\mathtt{T2,S_2}}}_{p=1}, \underbrace{x_t^{\mathtt{T,AR}}}_{}]^\intercal$$



$$\mathbf{x}_t^{\mathtt{D}} = [\underbrace{x_t^{\mathtt{D,LL}}}_{}, \underbrace{x_t^{\mathtt{D,AR}}}_{}]^\intercal$$

$$\mathbf{x} = \begin{bmatrix} \mathbf{x}^{\mathtt{D}} \\ \mathbf{x}^{\mathtt{T}} \end{bmatrix} = [\overbrace{\underbrace{x_t^{\mathtt{D,LL}}}_{}, \underbrace{x_t^{\mathtt{D,AR}}}_{}}^{\mathbf{x}_t^{\mathtt{D}}}, \overbrace{\underbrace{x_t^{\mathtt{T,LL}}}_{}, \underbrace{x_t^{\mathtt{T1,S_1}}, x_t^{\mathtt{T1,S_2}}}_{}, \underbrace{x_t^{\mathtt{T2,S_1}}, x_t^{\mathtt{T2,S_2}}}_{}, \underbrace{x_t^{\mathtt{T,AR}}}_{}}^{\mathbf{x}_t^{\mathtt{T}}}]^\intercal$$

Hidden state variables & model matrices (T)

$$\mathbf{x}_t^{\mathtt{T}} = [\underbrace{x_t^{\mathtt{T,LL}}}_{\text{—}}, \underbrace{x_t^{\mathtt{T1,S_1}}, x_t^{\mathtt{T1,S_2}}}_{\sim}, \underbrace{x_t^{\mathtt{T2,S_1}}, x_t^{\mathtt{T2,S_2}}}_{\sim}, \underbrace{x_t^{\mathtt{T,AR}}}_{\sim}]^{\mathsf{T}}$$

$$\mathbf{A}^{\mathtt{T}} = \text{block diag}\left(1, \begin{bmatrix} \cos\omega^{\mathtt{T1}} & \sin\omega^{\mathtt{T1}} \\ -\sin\omega^{\mathtt{T1}} & \cos\omega^{\mathtt{T1}} \end{bmatrix}, \begin{bmatrix} \cos\omega^{\mathtt{T2}} & \sin\omega^{\mathtt{T2}} \\ -\sin\omega^{\mathtt{T2}} & \cos\omega^{\mathtt{T2}} \end{bmatrix}, \phi^{\mathtt{T,AR}}\right)$$

$$\mathbf{Q}^{\mathtt{T}} = \text{block diag}\left([0, 0, 0, 0, 0, \sigma^{\mathtt{T,AR}}]^2\right)$$

$$\mathbf{C}^{\mathtt{T}} = [1, 1, 0, 1, 0, 1]$$

$$\mathbf{R}^{\mathtt{T}} = (\sigma^{\mathtt{T,V}})^2$$

$$\boldsymbol{\theta}^{\mathtt{T}} = [\sigma^{\mathtt{T,AR}}, \sigma^{\mathtt{T,V}}, \phi^{\mathtt{T,AR}}]^{\mathsf{T}} \quad \text{(Unknown parameters)}$$

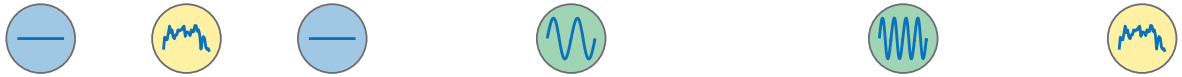
Hidden state variables & model matrices (D)

$$\mathbf{x}_t^{\mathtt{D}} = [\underbrace{x_t^{\mathtt{D,LL}}}_{\text{—}}, \underbrace{x_t^{\mathtt{D,AR}}}_{\text{〰}}]^\intercal$$

$$\begin{aligned}
\mathbf{A}^{\mathtt{D}} &= \text{block diag}\left(1, \phi^{\mathtt{D,AR}}\right) \\
\mathbf{Q}^{\mathtt{T}} &= \text{block diag}\left([0, \sigma^{\mathtt{D,AR}}]^2\right) \\
\mathbf{C}^{\mathtt{D}} &= [1, 1] \\
\mathbf{R}^{\mathtt{D}} &= (\sigma^{\mathtt{D,V}})^2 \\
\boldsymbol{\theta}^{\mathtt{D}} &= [\sigma^{\mathtt{D,AR}}, \sigma^{\mathtt{D,V}}, \phi^{\mathtt{D,AR}}]^\intercal \quad \text{(Unknown parameters)}
\end{aligned}$$

Hidden state variables & model matrices (D & T)

$$\mathbf{x}_t = [\overbrace{\underbrace{x_t^{\mathtt{D,LL}}}_{}, \underbrace{x_t^{\mathtt{D,AR}}}_{}}^{\mathbf{x}_t^{\mathtt{D}}}, \overbrace{\underbrace{x_t^{\mathtt{T,LL}}}_{}, \underbrace{x_t^{\mathtt{T1,S_1}}, x_t^{\mathtt{T1,S_2}}}_{}, \underbrace{x_t^{\mathtt{T2,S_1}}, x_t^{\mathtt{T2,S_2}}}_{}, \underbrace{x_t^{\mathtt{T,AR}}}_{}}^{\mathbf{x}_t^{\mathtt{T}}}]^\intercal$$



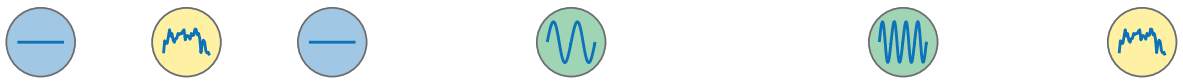
$$\begin{aligned}
\mathbf{A} &= \text{block diag}\left(\mathbf{A}^{\mathtt{D}}, \mathbf{A}^{\mathtt{T}}\right) \\
\mathbf{C} &= \text{block diag}\left(\mathbf{C}^{\mathtt{D}}, \mathbf{C}^{\mathtt{T}}\right) \\
\mathbf{R} &= \text{block diag}\left(\mathbf{R}^{\mathtt{D}}, \mathbf{R}^{\mathtt{T}}\right) \\
\mathbf{Q} &= \text{block diag}\left(\mathbf{Q}^{\mathtt{D}}, \mathbf{Q}^{\mathtt{T}}\right)
\end{aligned}$$

$$\mathbf{D} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} 1 & 1 & 0 & \phi^{\mathtt{D}|\mathtt{T}_1} & 0 & \phi^{\mathtt{D}|\mathtt{T}_2} & 0 & \phi^{\mathtt{D}|\mathtt{T}_2} \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

$$\begin{aligned}
\boldsymbol{\theta}^* &= [\ \phi^{\mathtt{D}|\mathtt{T}_1}, \ \phi^{\mathtt{D}|\mathtt{T}_2}, \ \phi^{\mathtt{D,AR}}, \ \phi^{\mathtt{T,AR}}, \ \sigma^{\mathtt{D,AR}}, \ \sigma^{\mathtt{T,AR}}, \ \sigma^{\mathtt{D,V}}, \ \sigma^{\mathtt{T,V}}]^\intercal \\
&= [-0.017, \ 0.01, \ 0.982, \ 0.997, \ 0.024, \ 0.28, \ 10^{-5}, \ 10^{-4}]^\intercal
\end{aligned}$$

Initial hidden state mean and covariance

$$\mathbf{x}_t = [\overbrace{\underbrace{x_t^{\mathtt{D,LL}}}_{}, \underbrace{x_t^{\mathtt{D,AR}}}_{}}^{\mathbf{x}_t^{\mathtt{D}}}, \overbrace{\underbrace{x_t^{\mathtt{T,LL}}}_{}, \underbrace{x_t^{\mathtt{T1,S_1}}, x_t^{\mathtt{T1,S_2}}}_{}, \underbrace{x_t^{\mathtt{T2,S_1}}, x_t^{\mathtt{T2,S_2}}}_{}, \underbrace{x_t^{\mathtt{T,AR}}}_{}}^{\mathbf{x}_t^{\mathtt{T}}}]^\intercal$$



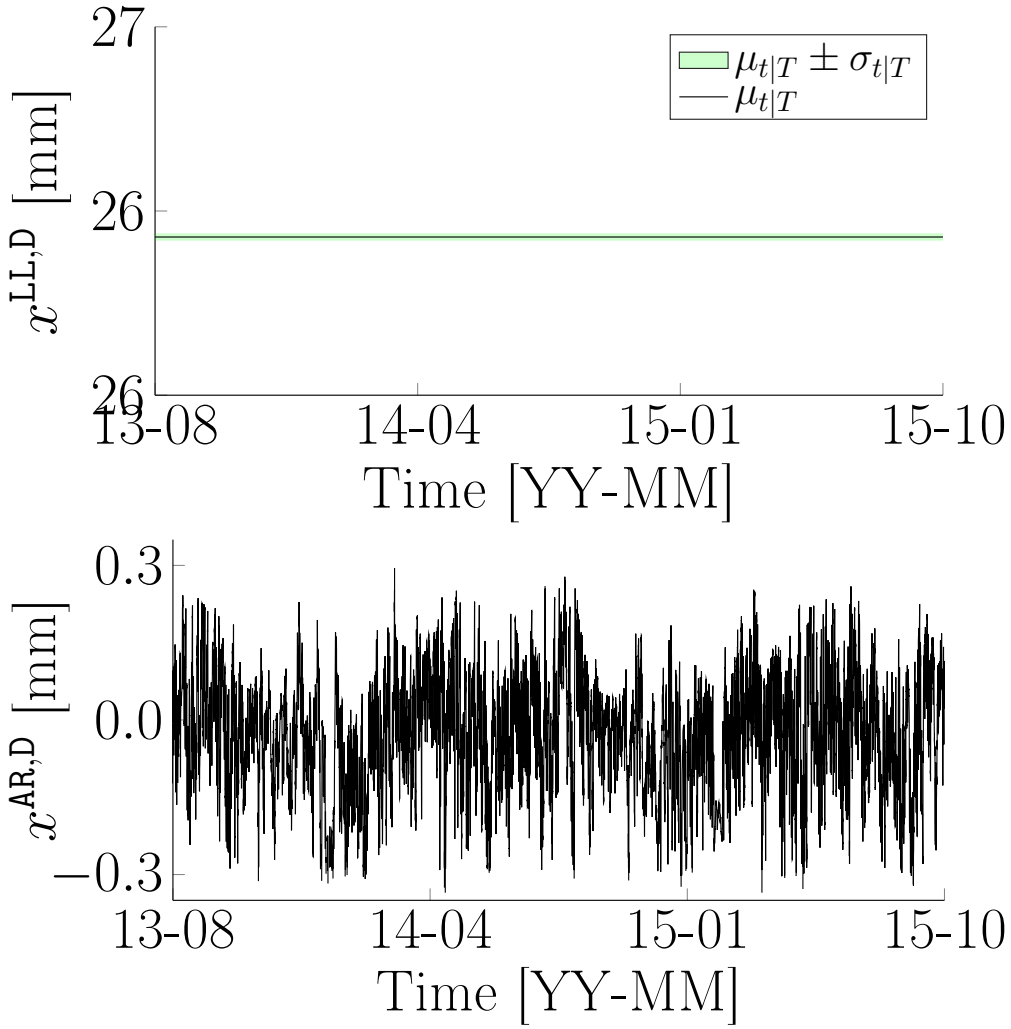
$$\mathbf{X}_0 \sim \mathcal{N}(\mathbf{x}_0; \boldsymbol{\mu}_0, \boldsymbol{\Sigma}_0) \left\{ \begin{array}{lcl} \boldsymbol{\mu}_0 & = & [26, 0, 6, 0, 0, 0, 0, 0]^\intercal \\ \boldsymbol{\Sigma}_0 & = & \mathrm{diag}([0.5, 0.1, 5, 10, 10, 10, 10, 1]^2) \end{array} \right.$$

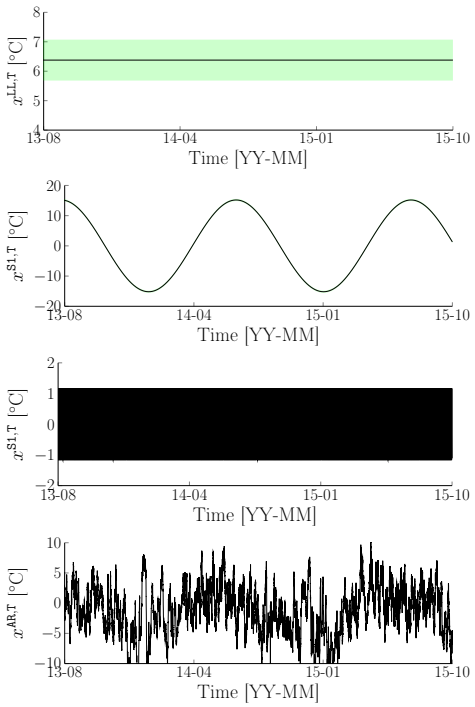
Hidden state estimation

$$\underbrace{\left.\begin{matrix}\mathbf{A}\\ \mathbf{C}\\ \mathbf{Q}\\ \mathbf{R}\end{matrix}\right\}\begin{aligned}\mathbf{x}_t &= \mathbf{A}\mathbf{x}_{t-1} + \underbrace{\mathbf{w}_t}_{W\sim\mathcal{N}(\mathbf{0},\mathbf{Q})}\\ y_t &= \mathbf{C}\mathbf{x}_t + \underbrace{\mathbf{v}_t}_{V\sim\mathcal{N}(\mathbf{0},\mathbf{R})}\end{aligned}}_{\text{Kalman Filter}}$$

$$(\boldsymbol{\mu}_{t|t}, \boldsymbol{\Sigma}_{t|t}, \mathcal{L}_t) = \text{Filter}(\boldsymbol{\mu}_{t-1|t-1}, \boldsymbol{\Sigma}_{t-1|t-1}, y_t; \mathbf{A}, \mathbf{C}, \mathbf{Q}, \mathbf{R})$$

Hidden state estimation (cont.)





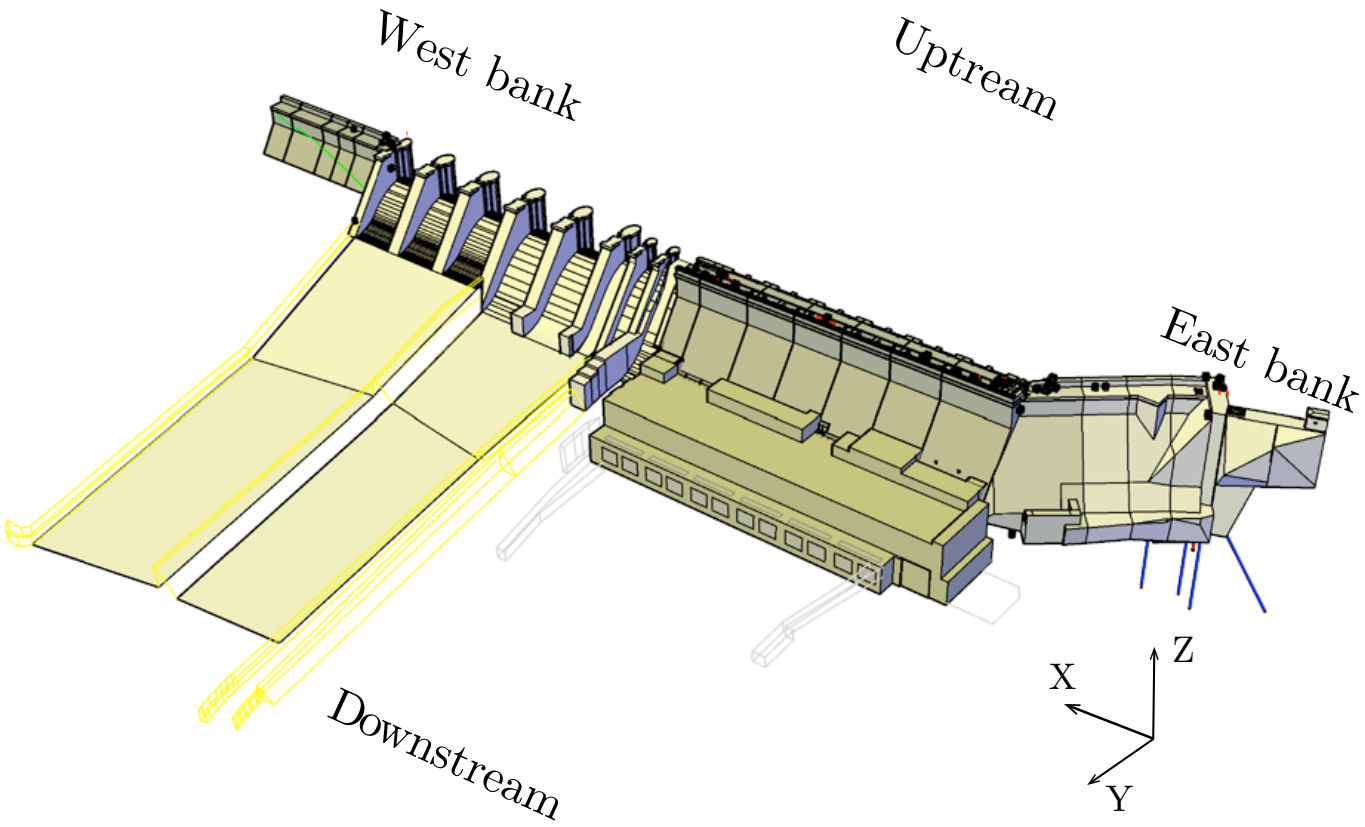
Section Outline

Case-Study SKF

4.1 Example : Dam in Canada

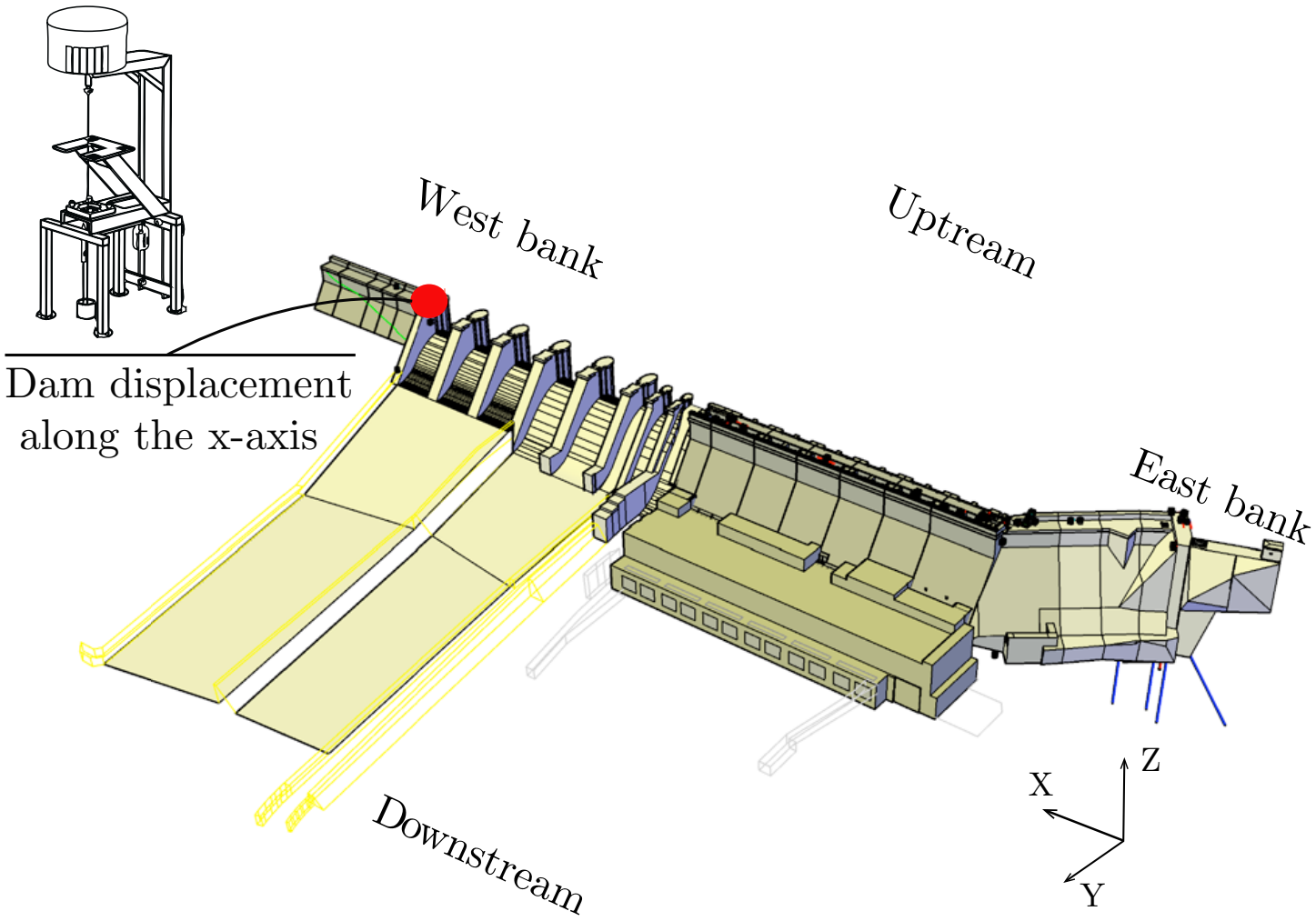
4.2 Results

Example : A dam in Canada



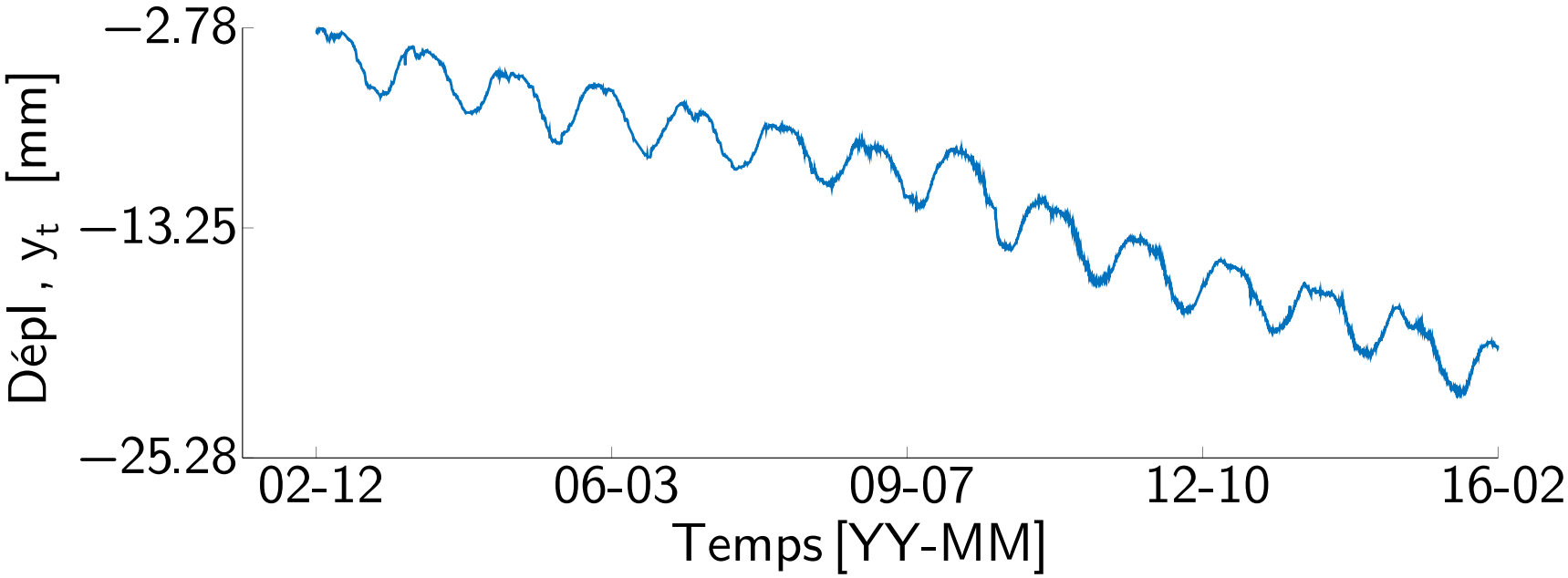
[Hydro Quebec]

Example : A dam in Canada



[Hydro Quebec]

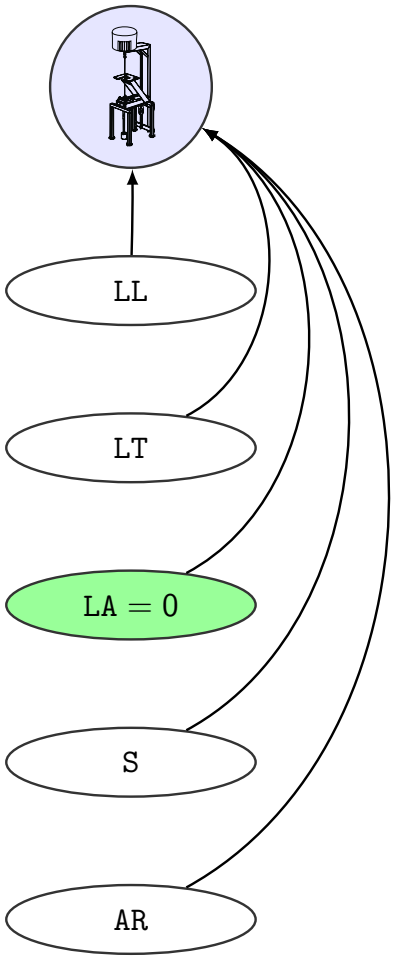
Data



[L.H. Nguyen 2017]

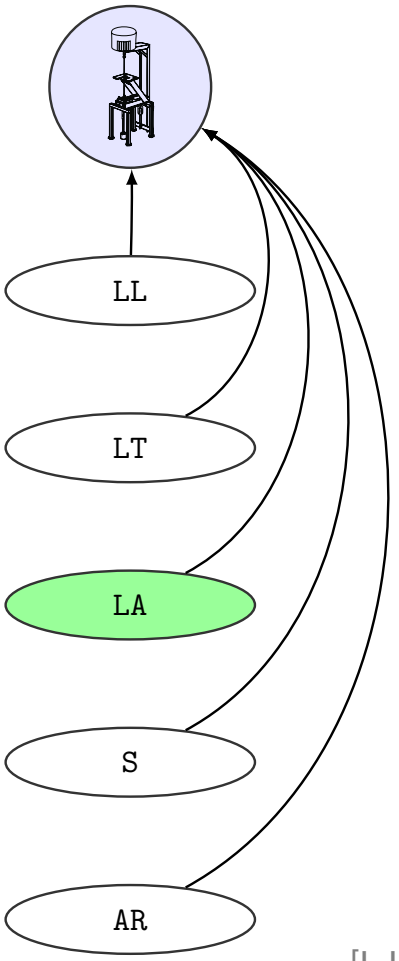
Model Construction

Normal



V.S.

Abnormal



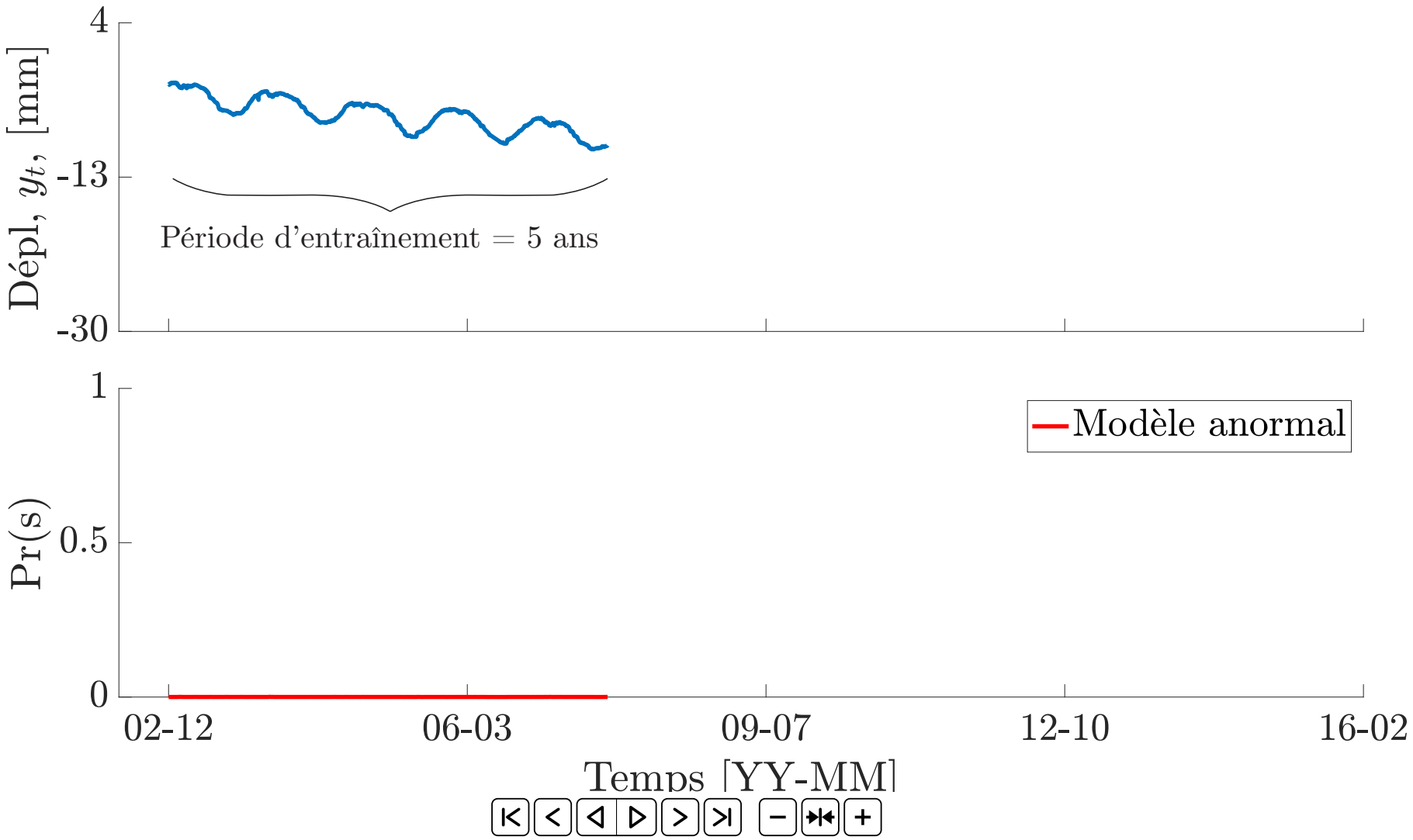
[L.H. Nguyen 2017]

Hidden state variables & Model matrices

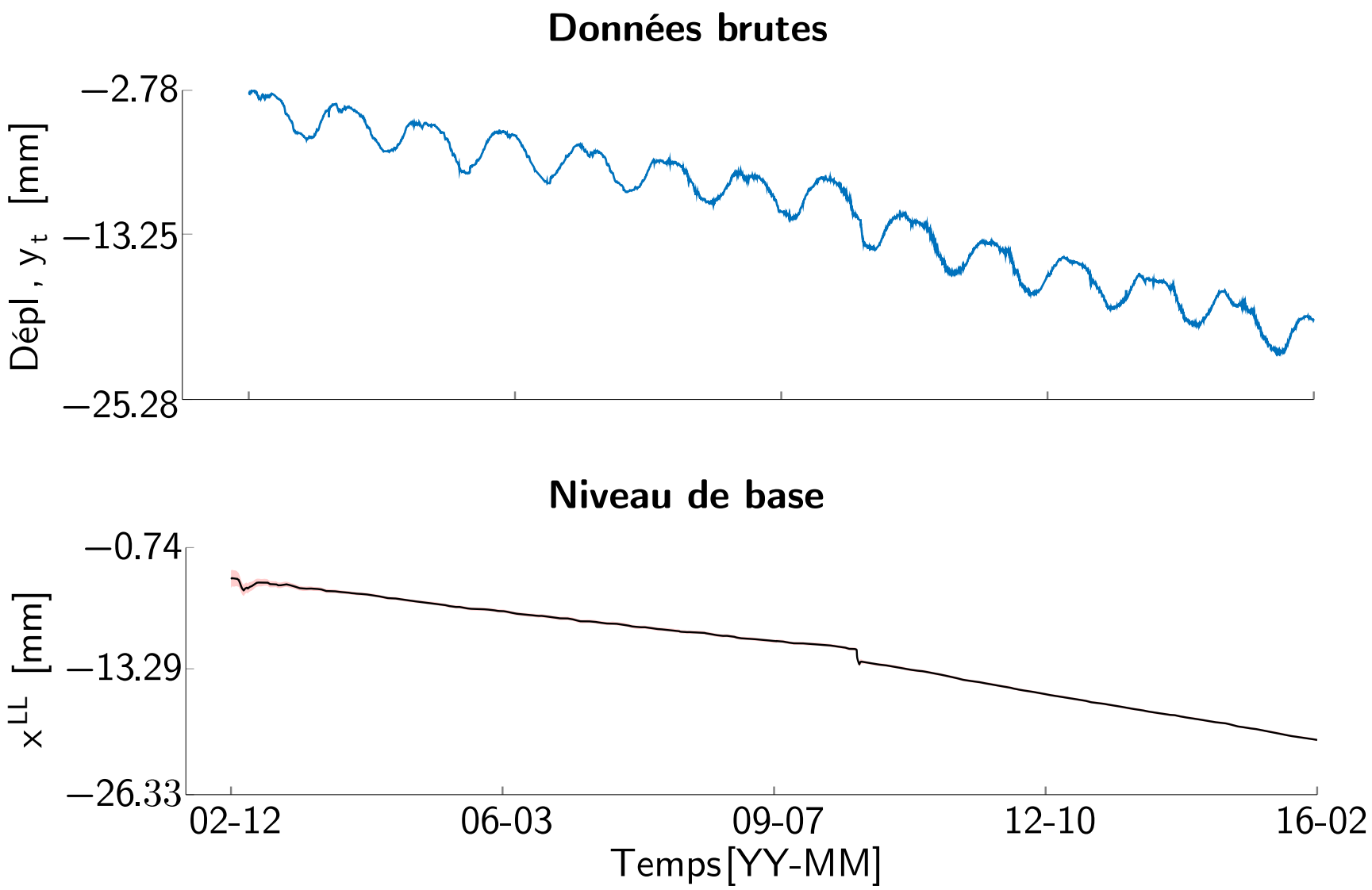
$$\mathbf{x}_t = [\underbrace{x_t^{\mathtt{L}}}_{}, \underbrace{x_t^{\mathtt{T}}}_{}, \underbrace{x_t^{\mathtt{LA}}}_{}, \underbrace{x_t^{\mathtt{T1,S_1}}, x_t^{\mathtt{T1,S_2}}}_{p=365.24}, \underbrace{x_t^{\mathtt{T2,S_1}}, x_t^{\mathtt{T2,S_2}}}_{p=182.62}, \underbrace{x_t^{\mathtt{AR}}}_{}]^{\mathsf{T}}$$

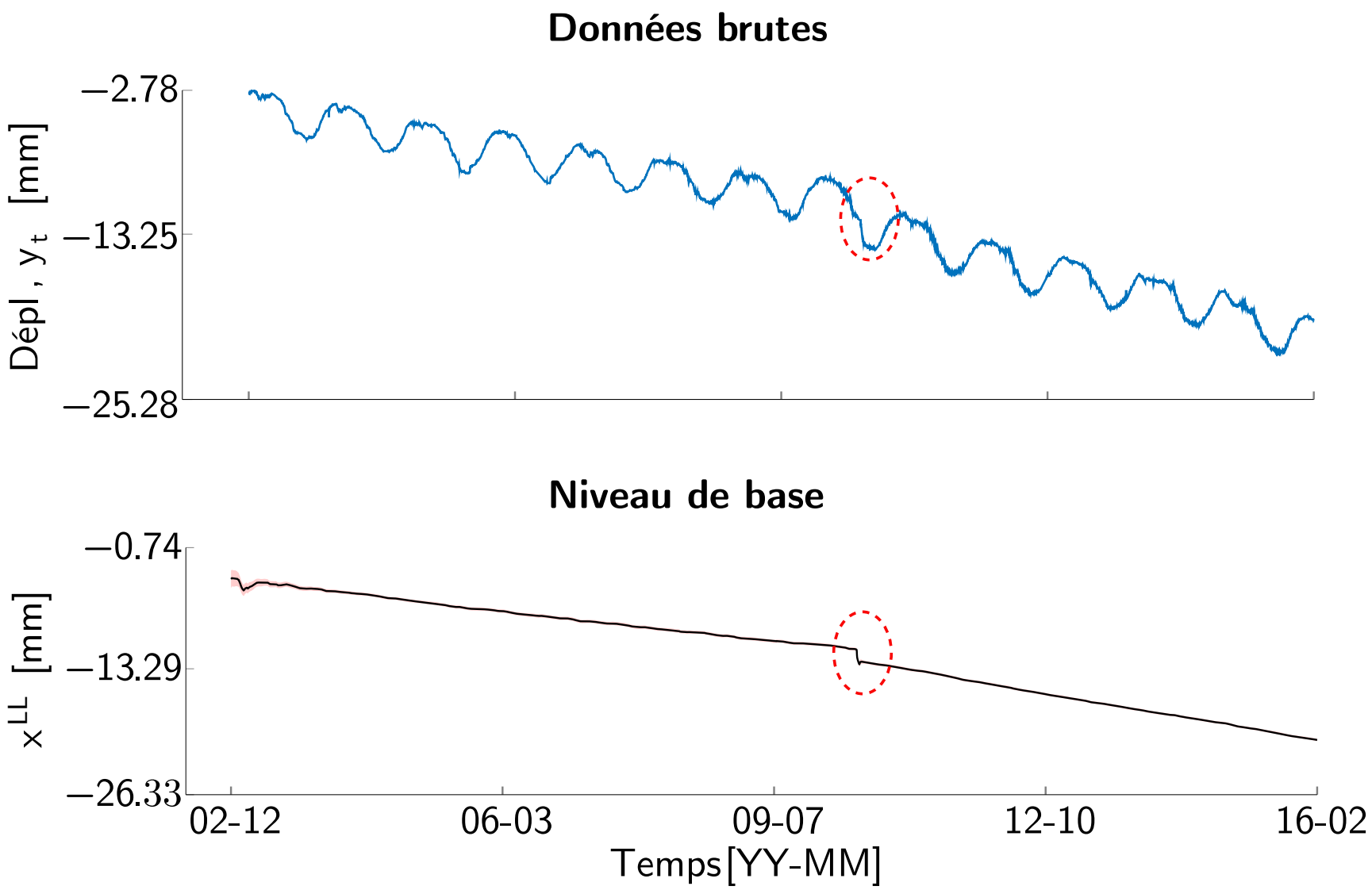
$$\mathbf{A}^{(1),\mathtt{LA}} = \begin{bmatrix} 1 & \Delta t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{A}^{(2),\mathtt{LA}} = \begin{bmatrix} 1 & \Delta t & \frac{\Delta t^2}{2} \\ 0 & 1 & \Delta t \\ 0 & 0 & 1 \end{bmatrix}$$

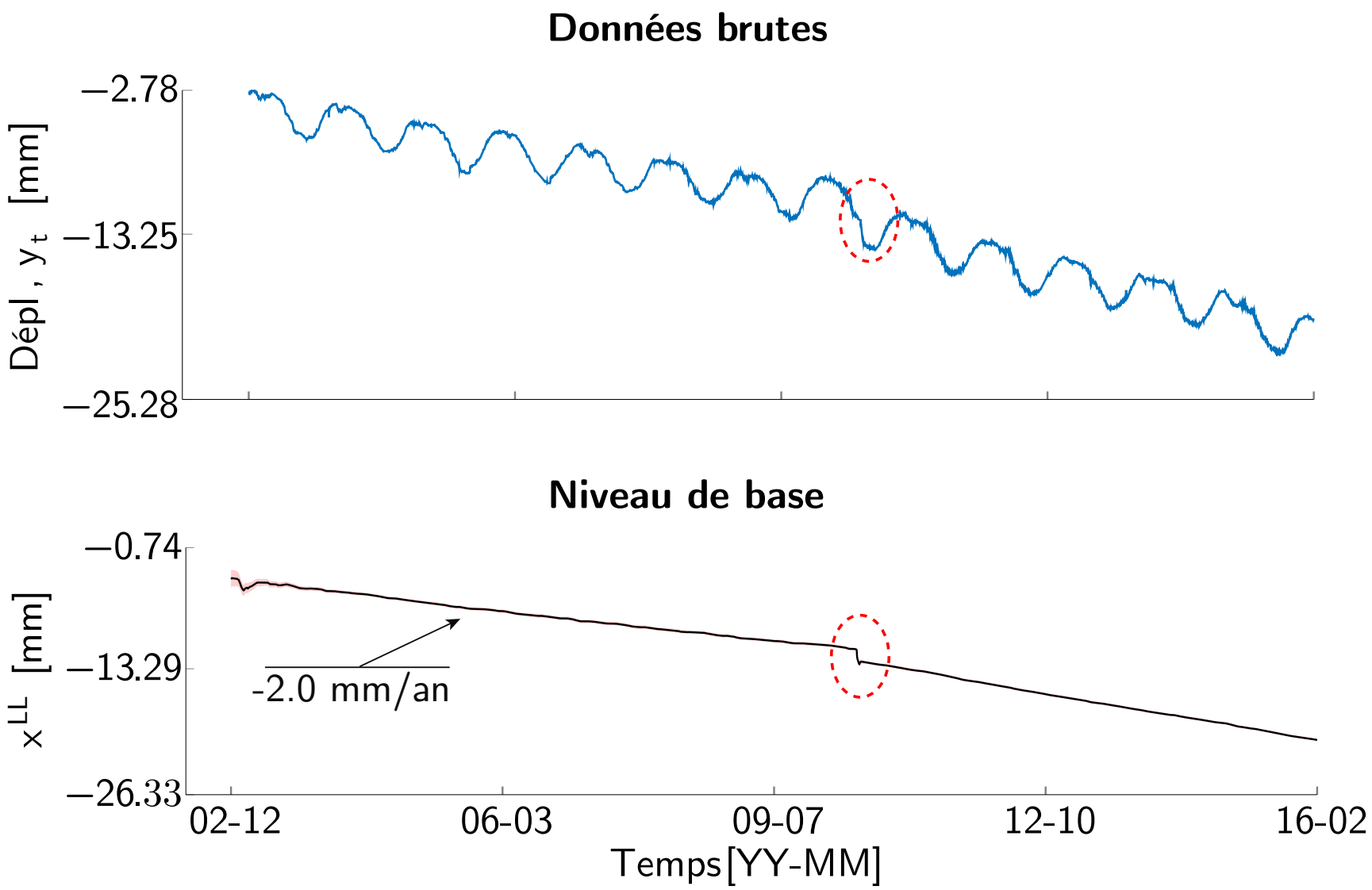
$$\mathbf{Z} = \begin{bmatrix} z^{11} & 1 - z^{11} \\ 1 - z^{22} & z^{22} \end{bmatrix}$$

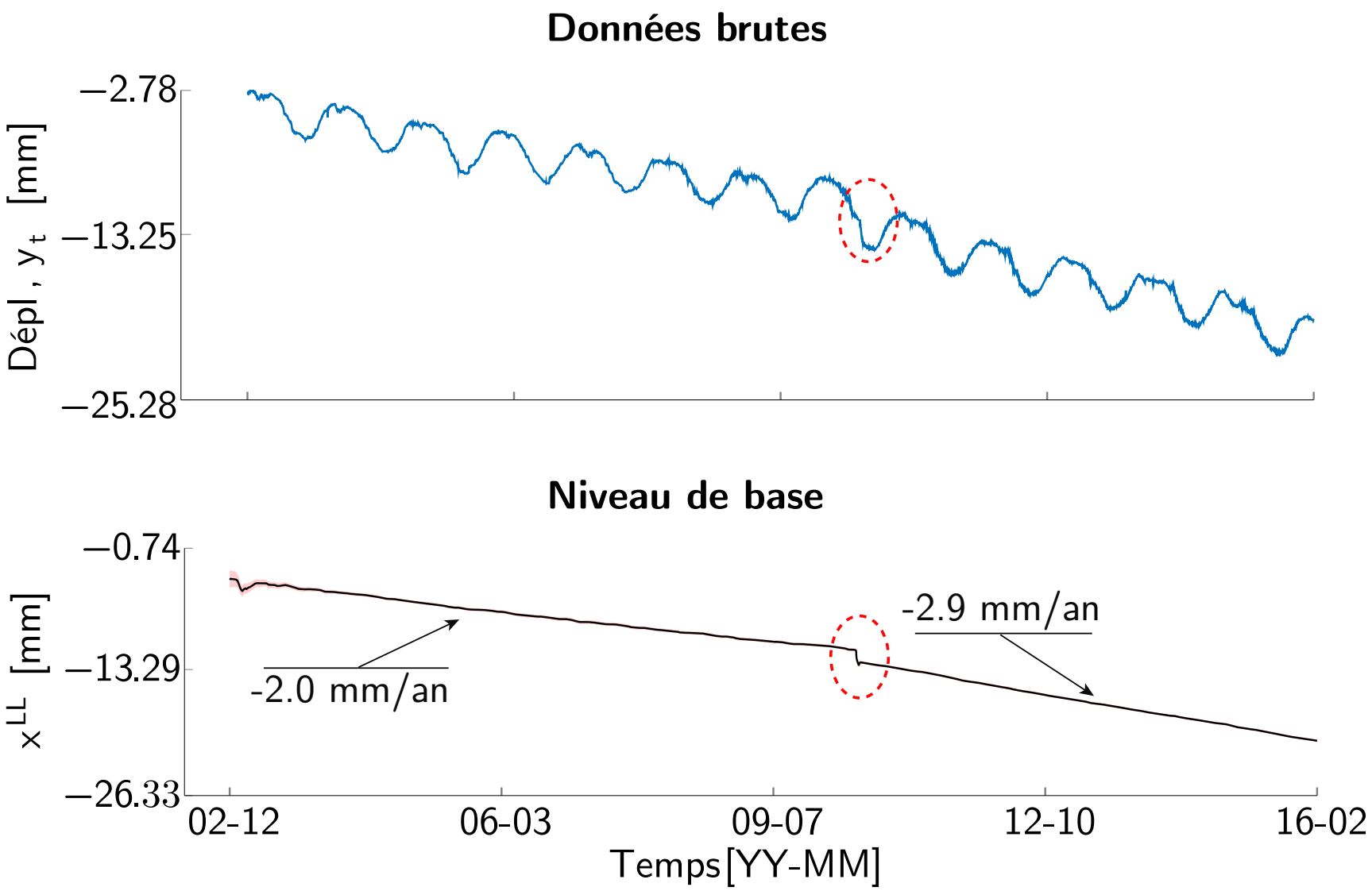


[L.H. Nguyen 2017]

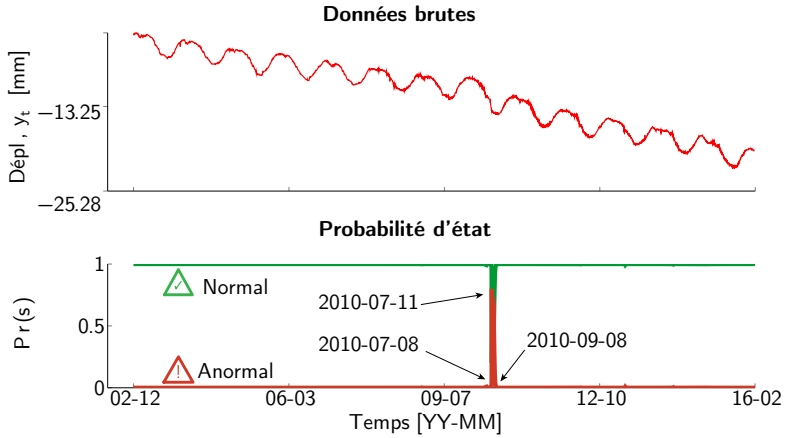

Données brutes
−2.78
−13.25
−25.28
Dépl , y_t [mm]
Niveau de base
−0.74
−13.29
−26.33
x^{LL} [mm]
02-12
06-03
09-07
12-10
16-02
Temps [YY-MM]

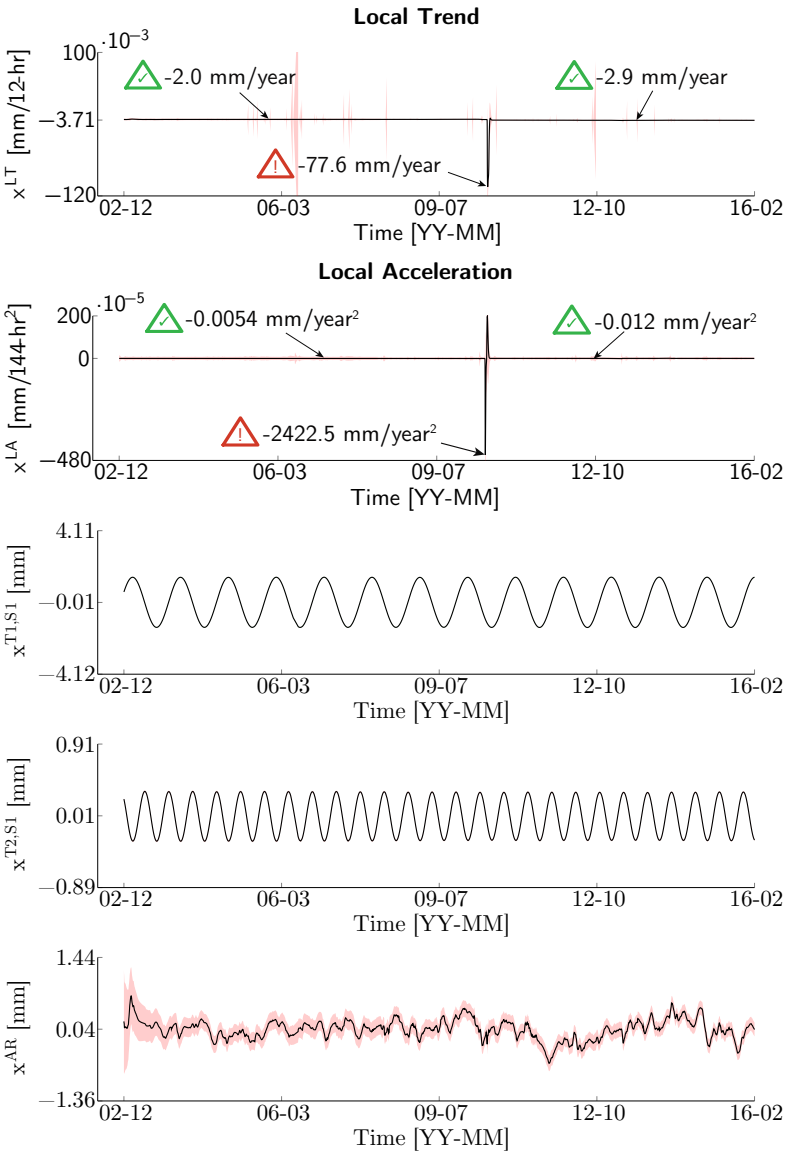

Données brutes
−2.78
−13.25
−25.28
Dépl , y_t [mm]
Niveau de base
−0.74
−13.29
−26.33
x^{LL} [mm]
02-12
06-03
09-07
12-10
16-02
Temps [YY-MM]


Données brutes
−2.78
−13.25
−25.28
Dépl , y_t [mm]
Niveau de base
−0.74
−13.29
−26.33
x^{LL} [mm]
-2.0 mm/an
02-12
06-03
09-07
12-10
16-02
Temps [YY-MM]


Données brutes
−2.78
−13.25
−25.28
Dépl , y_t [mm]
Niveau de base
−0.74
−13.29
−26.33
x^{LL} [mm]
-2.0 mm/an
-2.9 mm/an
02-12
06-03
09-07
12-10
16-02
Temps [YY-MM]

Results


Données brutes
Dépl., y_t [mm]
−13.25
−25.28
Probabilité d'état
Pr(s)
1
0.5
0
Normal
Anormal
2010-07-11
2010-07-08
2010-09-08
02-12
06-03
09-07
12-10
16-02
Temps [YY-MM]


Local Trend
x^{LT} [mm/12-hr]
$\cdot 10^{-3}$
100
−3.71
−120
-2.0 mm/year
-2.9 mm/year
-77.6 mm/year
02-12
06-03
09-07
12-10
16-02
Time [YY-MM]
Local Acceleration
x^{LA} [mm/144-hr^2]
$\cdot 10^{-5}$
200
0
−480
-0.0054 mm/year^2
-0.012 mm/year^2
-2422.5 mm/year^2
$x^{T1,S1}$ [mm]
4.11
−0.01
−4.12
$x^{T2,S1}$ [mm]
0.91
0.01
−0.89
x^{AR} [mm]
1.44
0.04
−1.36

Section Outline

Canari

5.1 Data preprocessing
5.2 Components
5.3 Examples of configuration files for Canari
5.4 Practical Examples
5.6 Other Libraries

Canari – Python library

canari

Data processing

- Loading data
- Normalization
- Resampling
- Train/val./test split

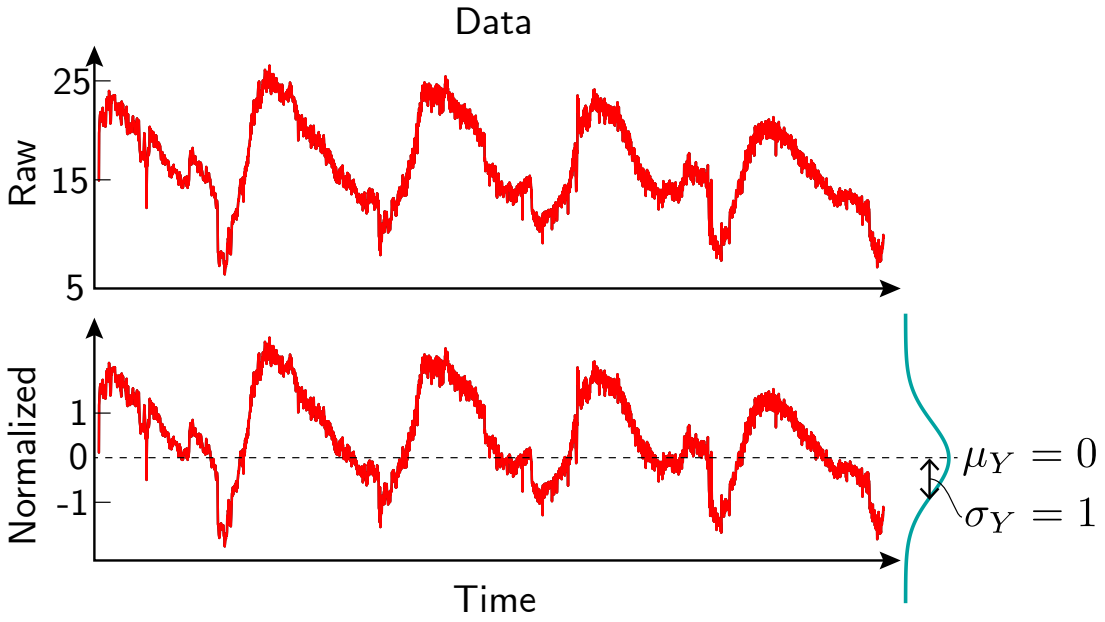
Timestamps: YYYY-MM-DD HH:MM:SS
Values: `float`

Raw → Normalized → Analyses → Unnormalization

Data processing

Timestamps: YYYY-MM-DD HH:MM:SS
Values: `float`

- Loading data
- Normalization
- Resampling
- Train/val./test split

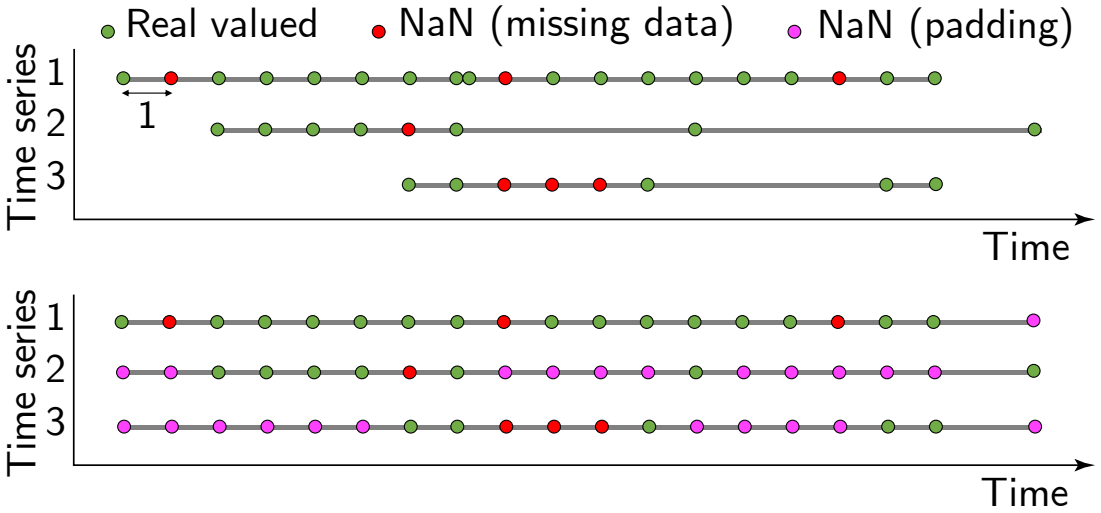


Raw → Normalized → Analyses → Unnormalization

Data processing

- Loading data
- Normalization
- Resampling
- Train/val./test split

Timestamps: YYYY-MM-DD HH:MM:SS
Values: `float`

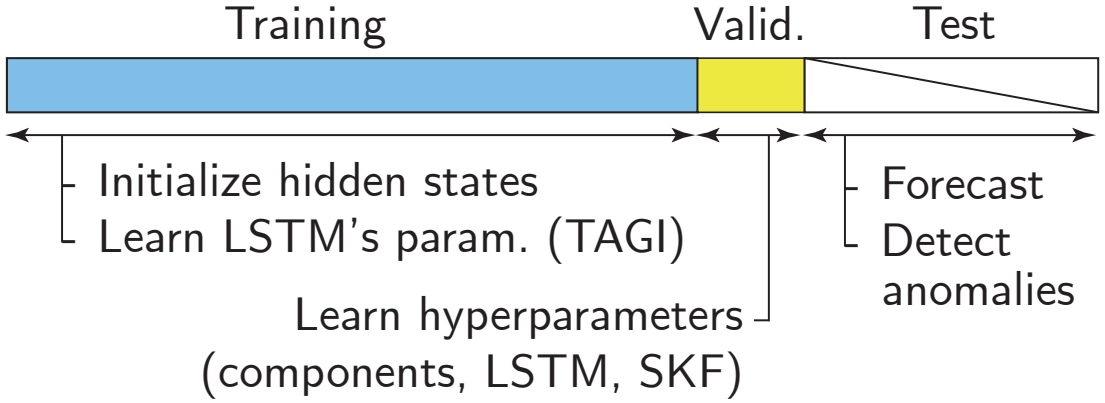


Raw → Normalized → Analyses → Unnormalization

Data processing

- Loading data
- Normalization
- Resampling
- Train/val./test split

Timestamps: YYYY-MM-DD HH:MM:SS
Values: `float`



Raw → Normalized → Analyses → Unnormalization

Canari's basic components

Baseline

- `LocalLevel( )`
- `LocalTrend( )`
- `LocalAcceleration( )`

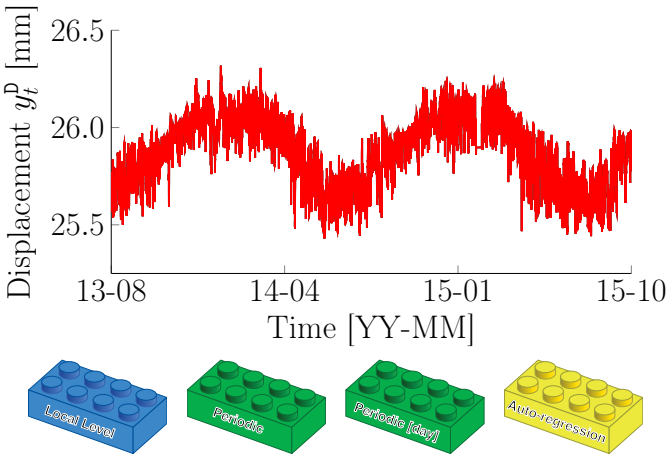
Periodic/Reccurent

- `Periodic( )`
- `LstmNetwork( )`

Residual

- `WhiteNoise( )`: $\sim V$
- `Autoregression( )`

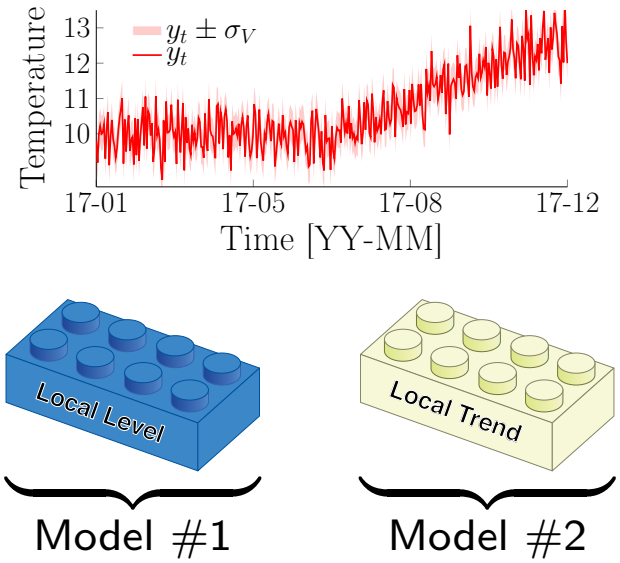
Displacement example




```
# Components
LL = LocalLevel(mu_states=[25.5],
                var_states=[1],
                std_error=0)
PD_y = Periodic(period=365.2422)
PD_d = Periodic(period=1)
AR = Autoregression(
                std_error=param["1"],
                phi=param["0.9"])
V = WhiteNoise(std_error=param["0.1"])

# Model building
model = Model(LL, PD_y, PD_d, AR, V)
```

`param["..."]` are optimized using the **Ray** library

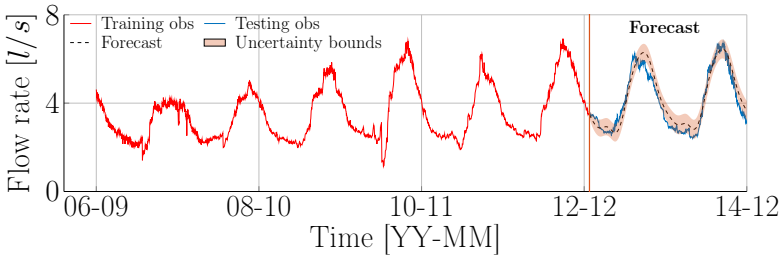
Regime Switching Example



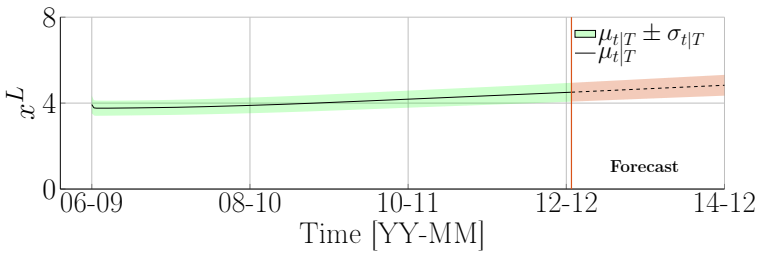
Configuration file

```
# Normal model
model = Model(LL, AR, V)

# Abnormal model
ab_model = Model(LT, AR, V)

# Switching Kalman filter
skf = SKF(
    norm_model=model,
    abnorm_model=ab_model,
    std_transition_error=1e-3,
    norm_to_abnorm_prob=1e-4,
    abnorm_to_norm_prob=1e-4,
    norm_model_prior_prob=0.99)
```

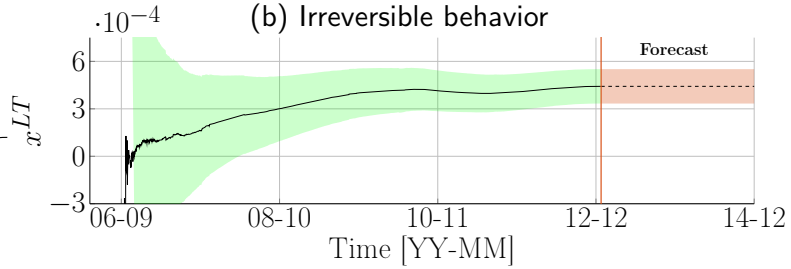
Water Flow in a Dam



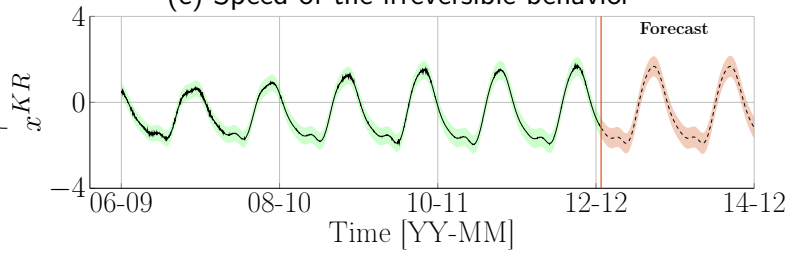
(a) Raw time series: Flow rate [l/s]



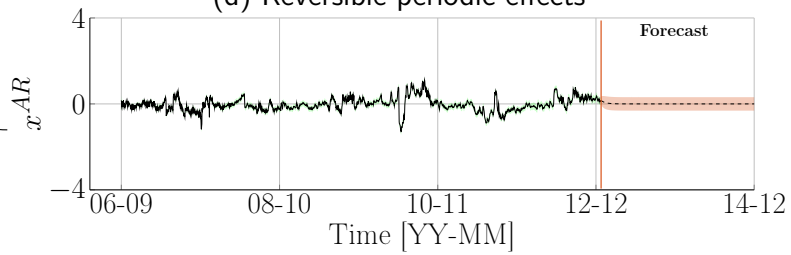
(b) Irreversible behavior



(c) Speed of the irreversible behavior

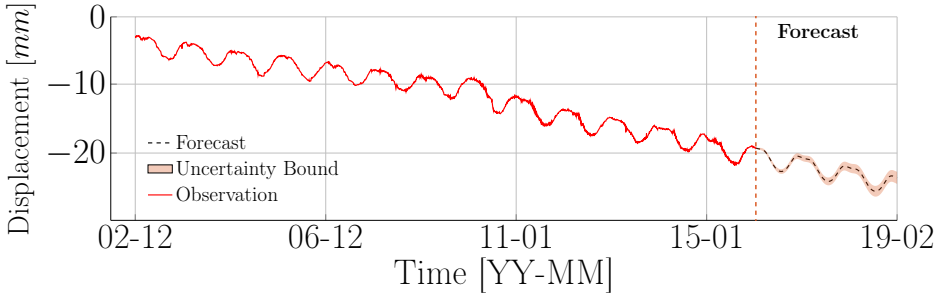


(d) Reversible periodic effects

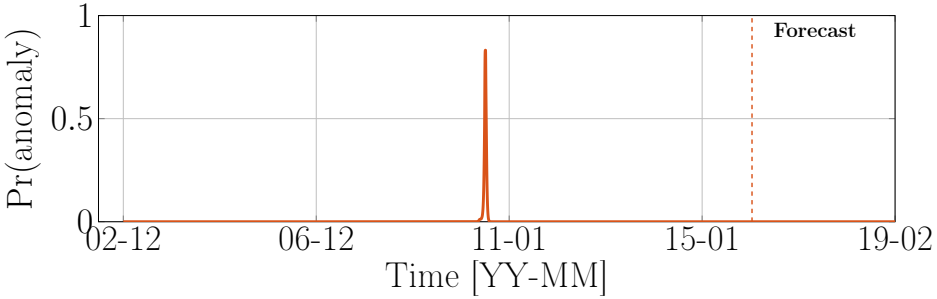


(e) Residual term that cannot be explained by the model

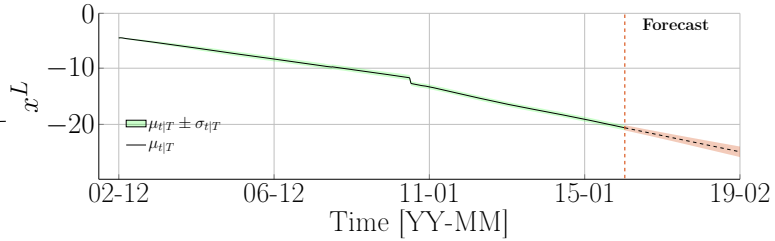
Anomaly Detection



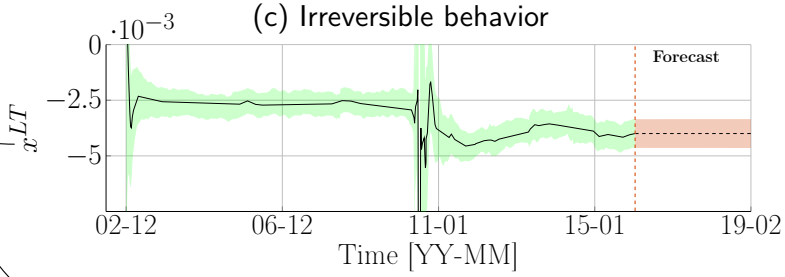
(a) Raw time series: displacement [*mm*]



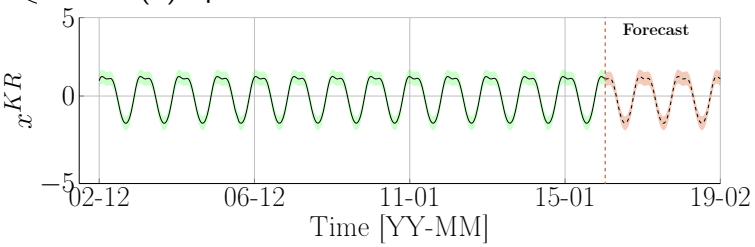
(b) Probability that a system is in an abnormal regime



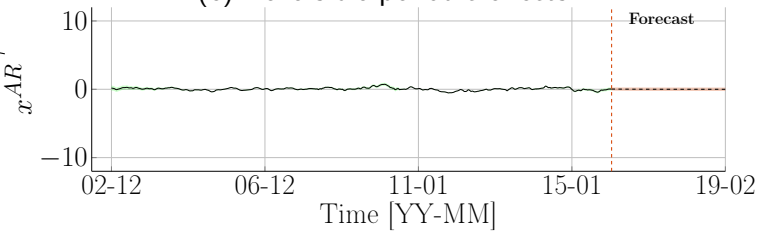
(c) Irreversible behavior



(d) Speed of the irreversible behavior

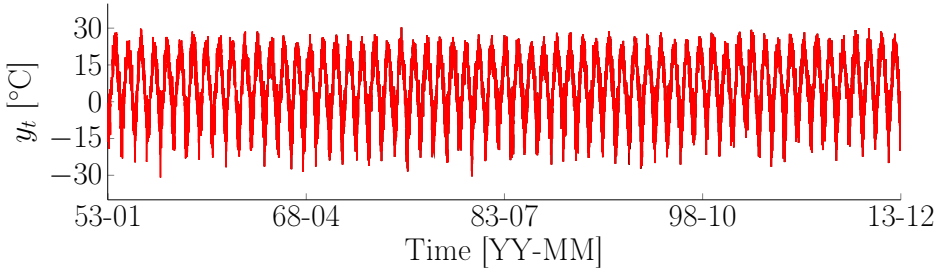


(e) Reversible periodic effects



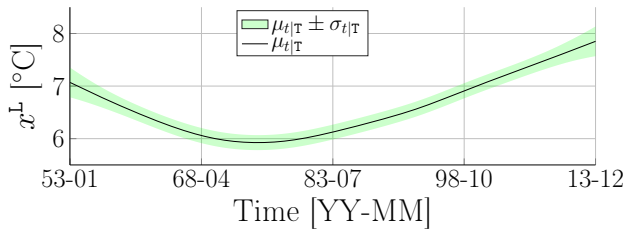
(f) Residual term that cannot be explained by the model

Quantifying the rate of change

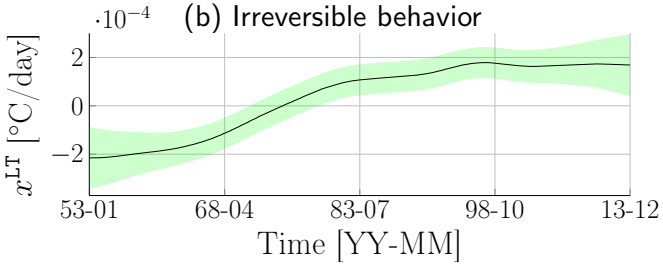


(a) Temperature YUL: Flow rate [°C]

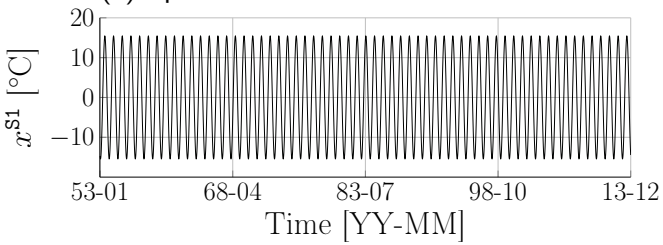
$$\mu^{\mathrm{LT}}_{T|T} = 0.17 \times 10^{-4} {}^{\circ}C/day \times 365 days = 0.06^{\circ}C/year$$
$$\sigma^{\mathrm{LT}}_{T|T} = 0.05^{\circ}C/year$$



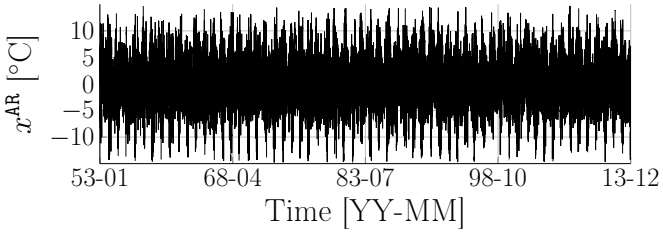
(b) Irreversible behavior



(c) Speed of the irreversible behavior



(d) Reversible periodic effects



(e) Residual term that cannot be explained by the model

canari

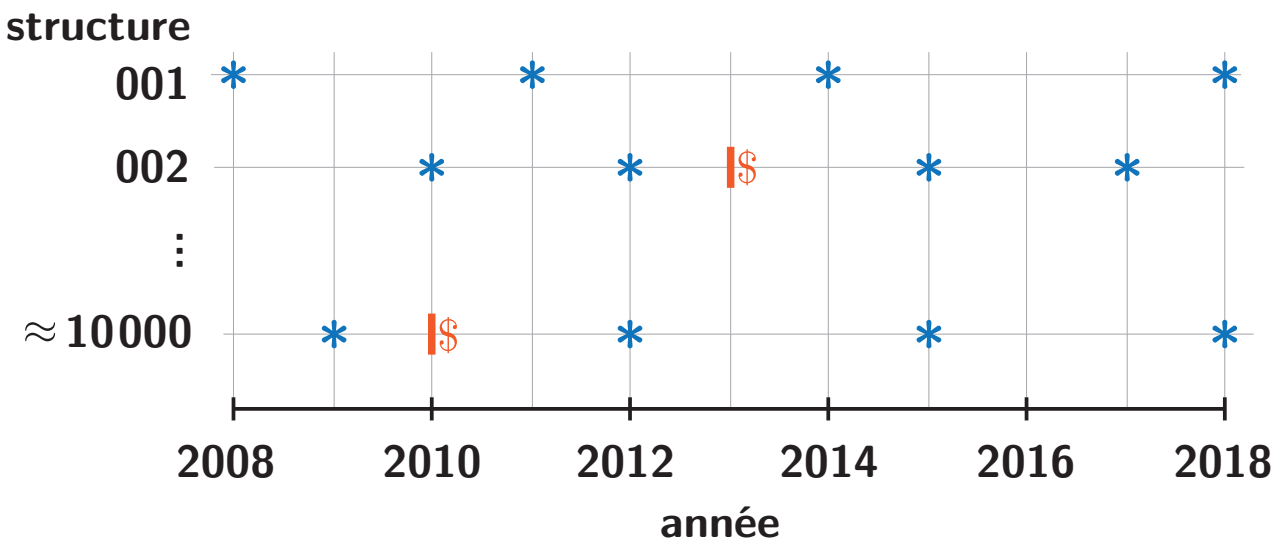
Canari release on **GitHub** → July 1, 2025

- Preprocessing tools: Data loader, normalization & time-synchronization
- Automated: model construction, parameter estimation & state estimation
- Post-processing tools: export + plots
- An open platform where you can develop **your own tools**

Other Python libraries for time-series forecasting

- **Dynamax** (K. Murphy @Google)
  - Low-level library for the Kalman filter and smoother & +++
  - General purpose library
  - Require a manual definition of models
- **Bayesian Structural Time Series/Causal Impact** (Google)
  - High-level library for time-series using the Kalman filter
  - Several generic models to analyze a large number of time series
  - Not best suited for the civil engineering reality...

Visual inspection data



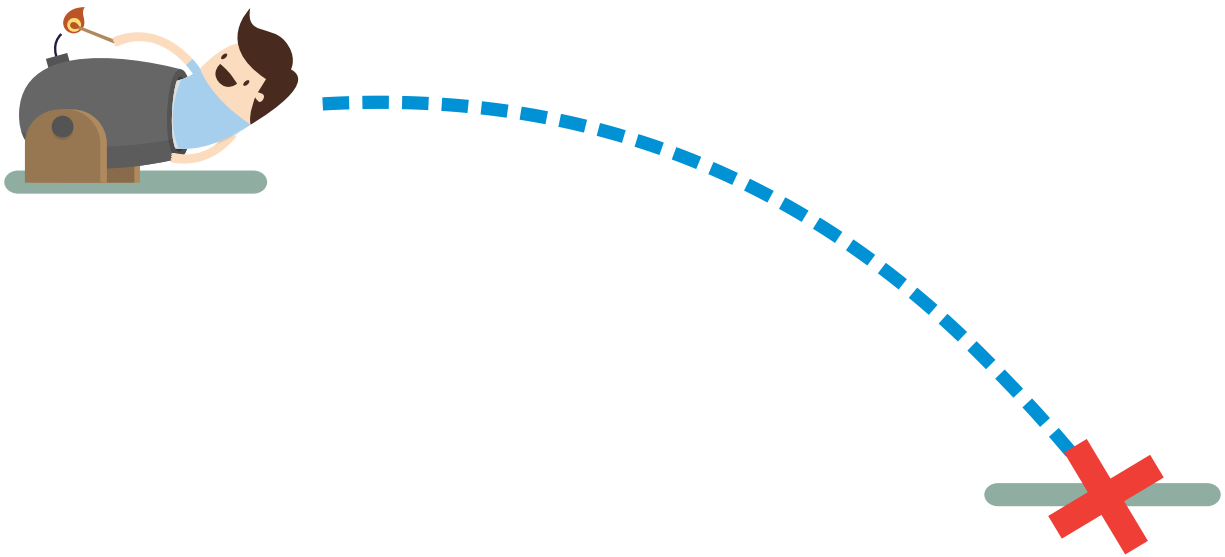
Year: 2017
Structure: #002

$$\left.\begin{array}{l} > 10^6 \\ \{1:5\} \end{array}\right\} \begin{array}{l} \text{data points} \\ \text{per structure} \end{array}$$

$$\mathbf{x}_{002} = \left\{ \begin{array}{c} \text{type} \\ \text{material} \\ \#\text{vehicule} \\ \text{location} \\ \vdots \end{array} \right\}$$

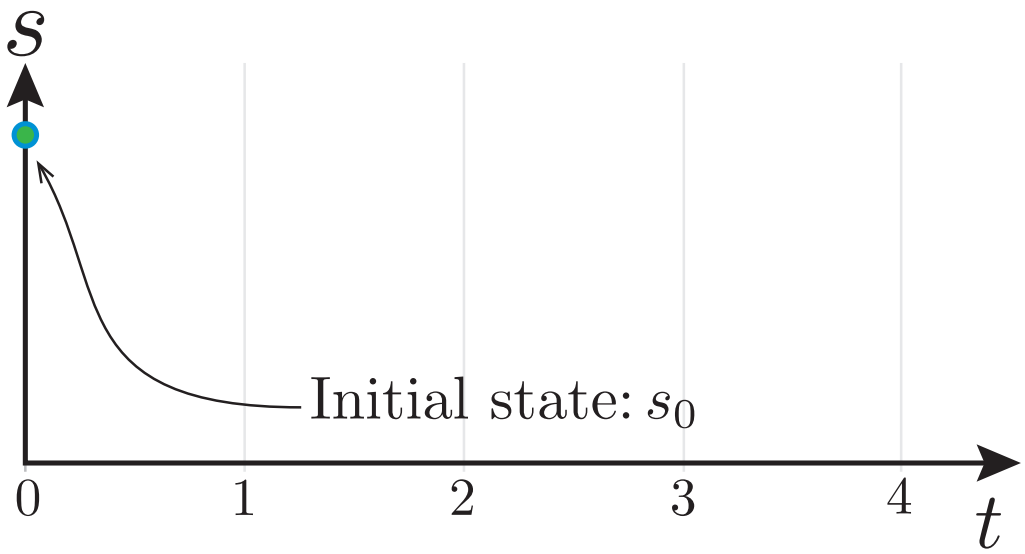
Model structure

$$\overbrace{\begin{aligned} y_1 = s_{t+1} + v &= s_t + \dot{s}_t \Delta t + \tfrac{1}{2}\ddot{s}_t \Delta t^2 + w + v && \text{(Position)} \\ \dot{s}_{t+1} &= \dot{s}_t + \ddot{s}_t \Delta t + \dot{w} && \text{(Speed)} \\ \ddot{s}_{t+1} &= \ddot{s}_t + \ddot{w} && \text{(Acceleration)} \end{aligned}}^{\text{Kinematic equations}}$$



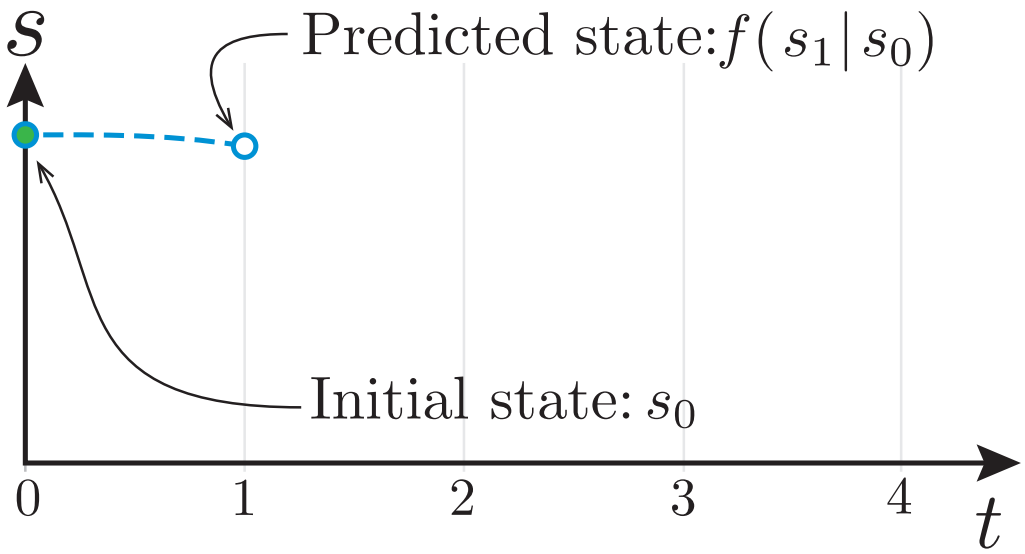
Model structure

$$\overbrace{\begin{aligned} y_1 = s_{t+1} + v &= s_t + \dot{s}_t \Delta t + \tfrac{1}{2}\ddot{s}_t \Delta t^2 + w + v && \text{(Position)} \\ \dot{s}_{t+1} &= \dot{s}_t + \ddot{s}_t \Delta t + \dot{w} && \text{(Speed)} \\ \ddot{s}_{t+1} &= \ddot{s}_t + \ddot{w} && \text{(Acceleration)} \end{aligned}}^{\text{Kinematic equations}}$$



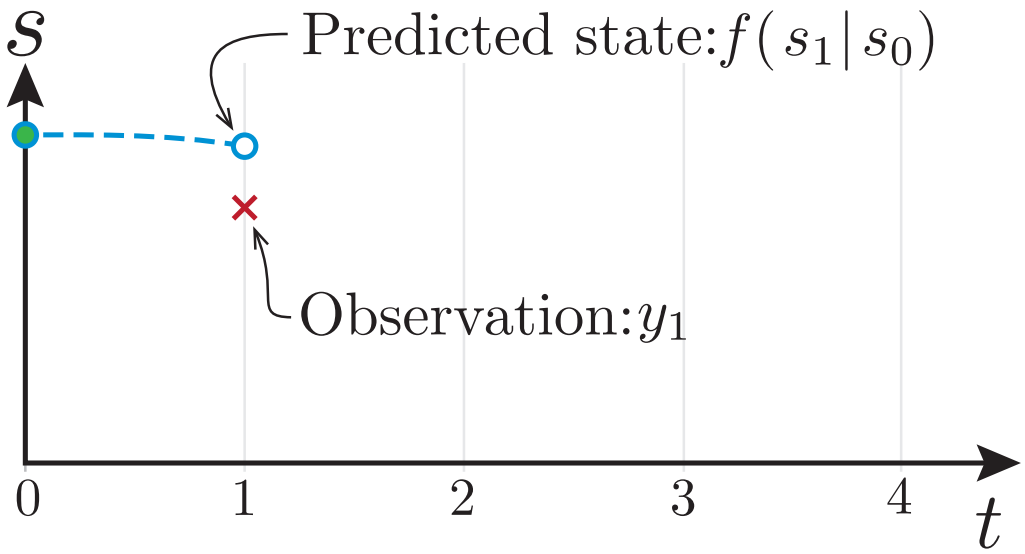
Model structure

$$\overbrace{\begin{aligned} y_1 = s_{t+1} + v &= s_t + \dot{s}_t \Delta t + \tfrac{1}{2}\ddot{s}_t \Delta t^2 + w + v && \text{(Position)} \\ \dot{s}_{t+1} &= \dot{s}_t + \ddot{s}_t \Delta t + \dot{w} && \text{(Speed)} \\ \ddot{s}_{t+1} &= \ddot{s}_t + \ddot{w} && \text{(Acceleration)} \end{aligned}}^{\text{Kinematic equations}}$$



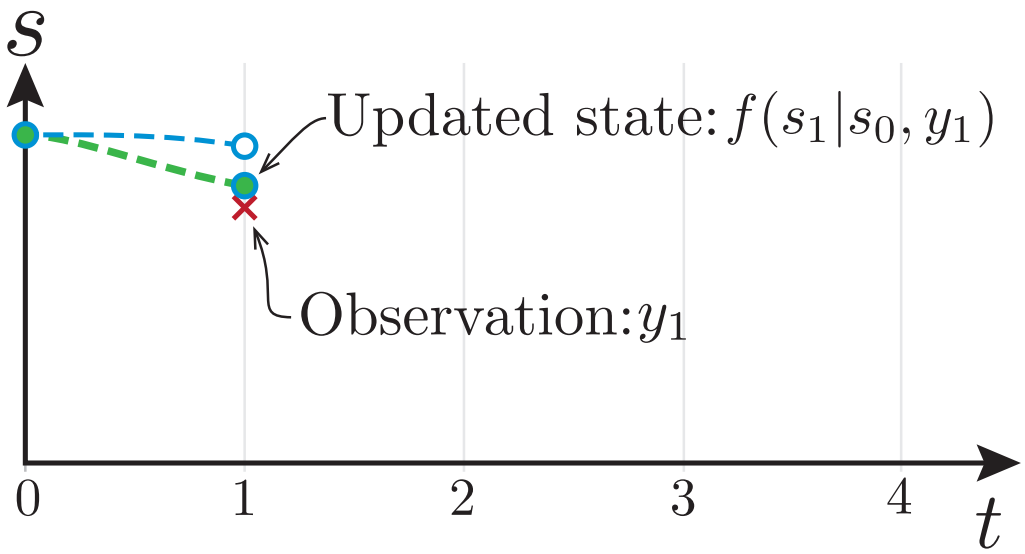
Model structure

$$
\overbrace{\begin{aligned}
y_1 = s_{t+1} + v &= s_t + \dot{s}_t \Delta t + \tfrac{1}{2}\ddot{s}_t \Delta t^2 + w + v && \text{(Position)} \\
\dot{s}_{t+1} &= \dot{s}_t + \ddot{s}_t \Delta t + \dot{w} && \text{(Speed)} \\
\ddot{s}_{t+1} &= \ddot{s}_t + \ddot{w} && \text{(Acceleration)}
\end{aligned}}^{\text{Kinematic equations}}
$$



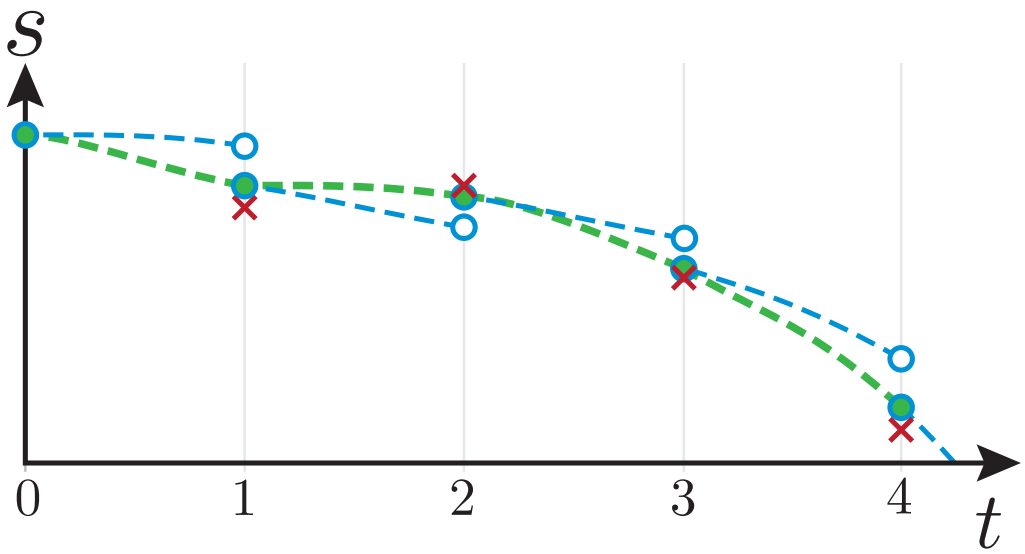
Model structure

$$\overbrace{\begin{aligned} y_1 = s_{t+1} + v &= s_t + \dot{s}_t \Delta t + \tfrac{1}{2}\ddot{s}_t \Delta t^2 + w + v && \text{(Position)} \\ \dot{s}_{t+1} &= \dot{s}_t + \ddot{s}_t \Delta t + \dot{w} && \text{(Speed)} \\ \ddot{s}_{t+1} &= \ddot{s}_t + \ddot{w} && \text{(Acceleration)} \end{aligned}}^{\text{Kinematic equations}}$$

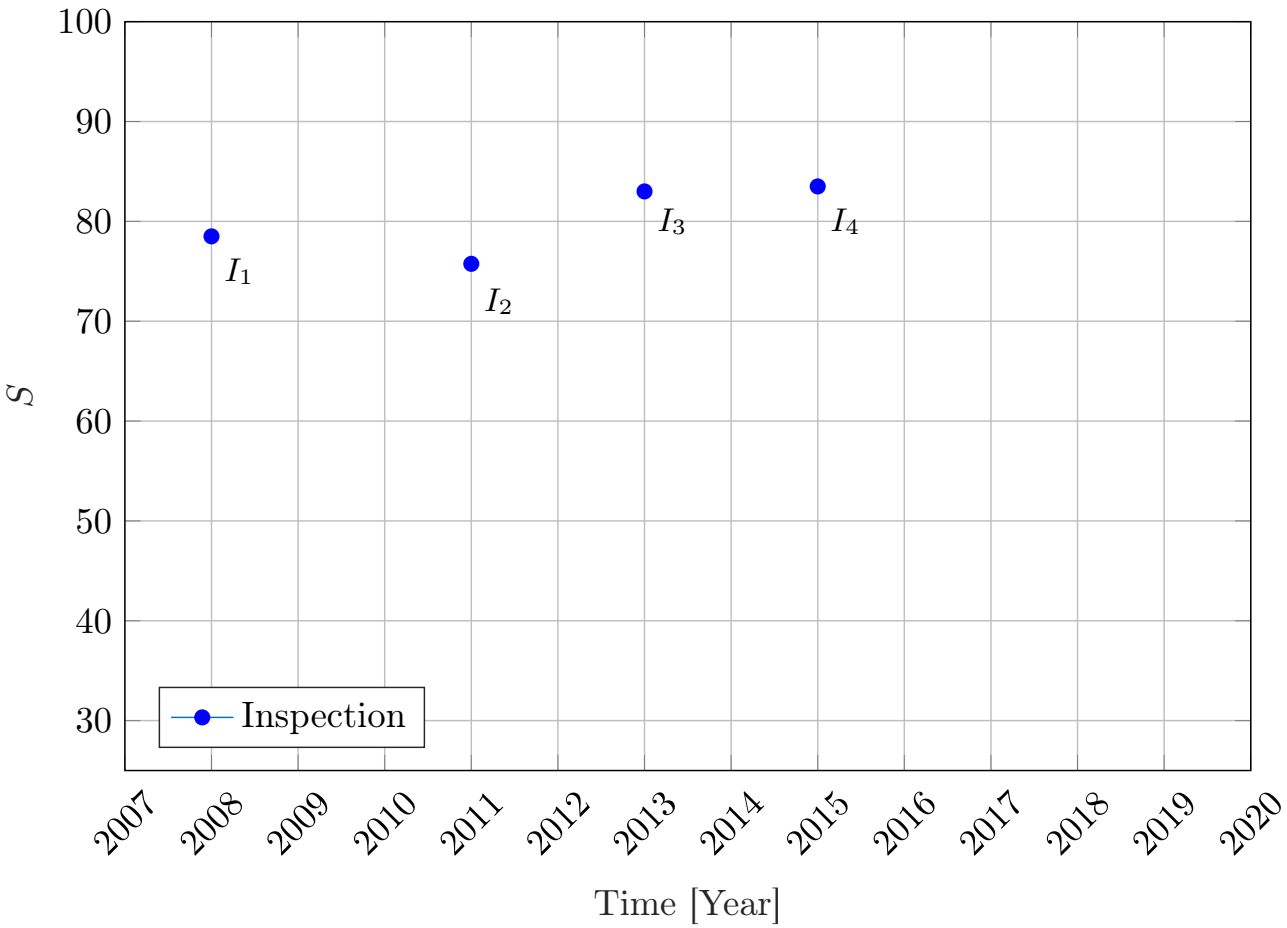


Model structure

$$\overbrace{\begin{aligned} y_1 = s_{t+1} + v &= s_t + \dot{s}_t \Delta t + \tfrac{1}{2}\ddot{s}_t \Delta t^2 + w + v && \text{(Position)} \\ \dot{s}_{t+1} &= \dot{s}_t + \ddot{s}_t \Delta t + \dot{w} && \text{(Speed)} \\ \ddot{s}_{t+1} &= \ddot{s}_t + \ddot{w} && \text{(Acceleration)} \end{aligned}}^{\text{Kinematic equations}}$$



Inspector's uncertainty



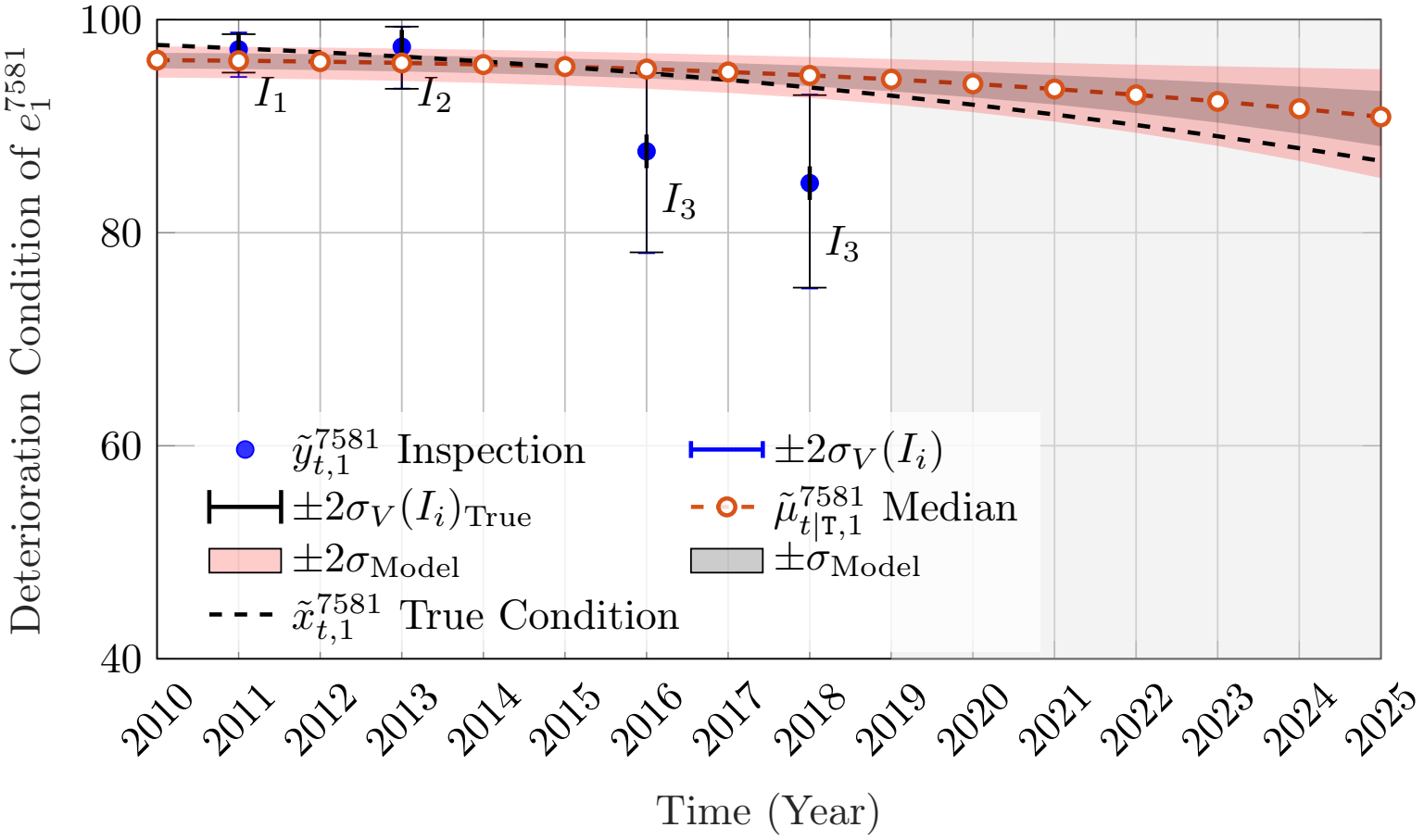
The uncertainty must be estimated for each inspector

[Google images]

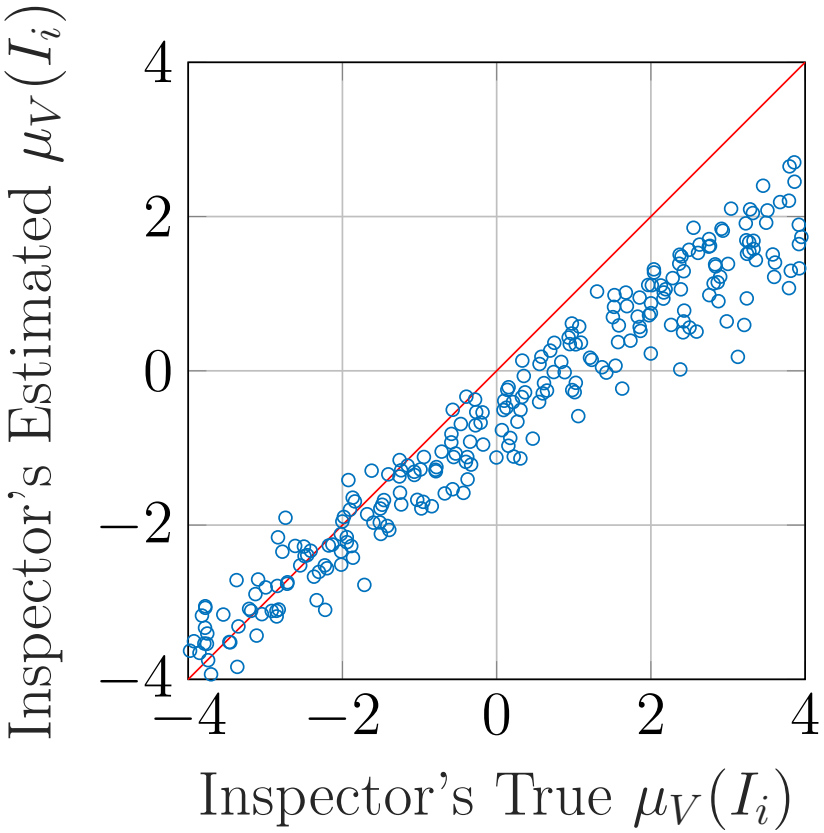
Synthetic data

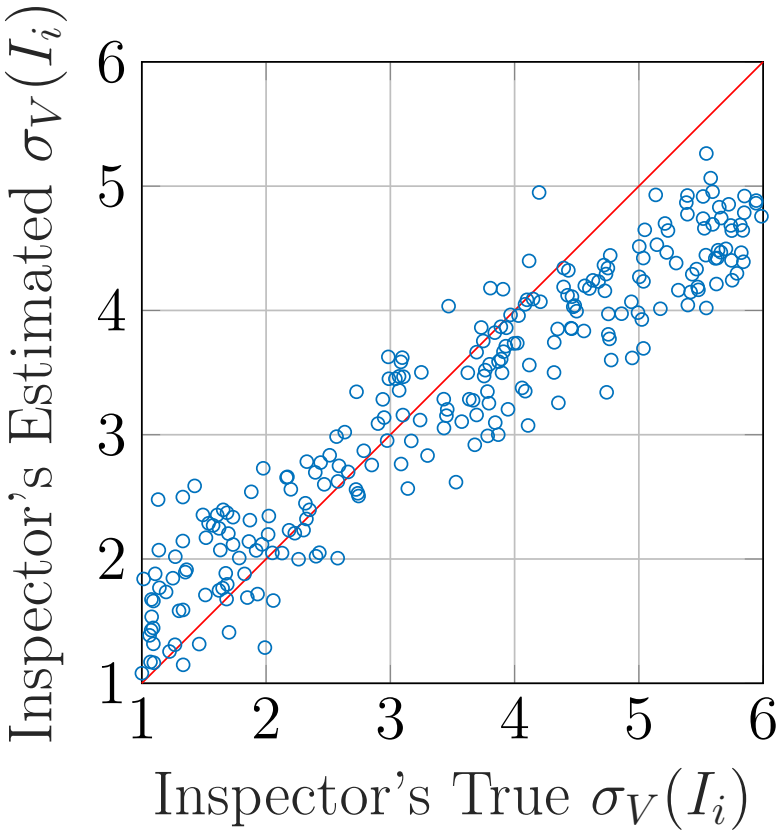
[Google images]

Example – Synthetic data



Validation set – Synthetic data

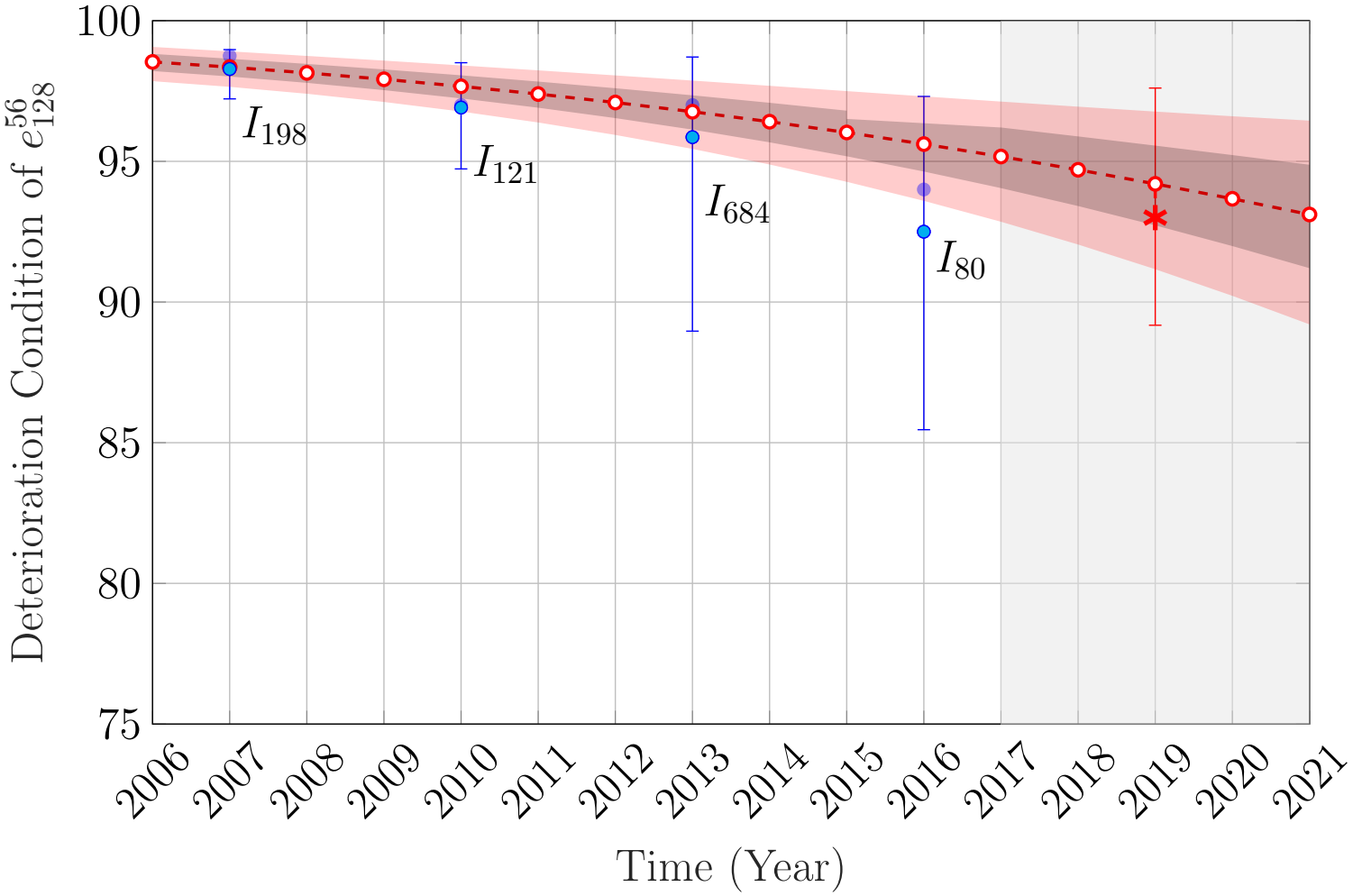




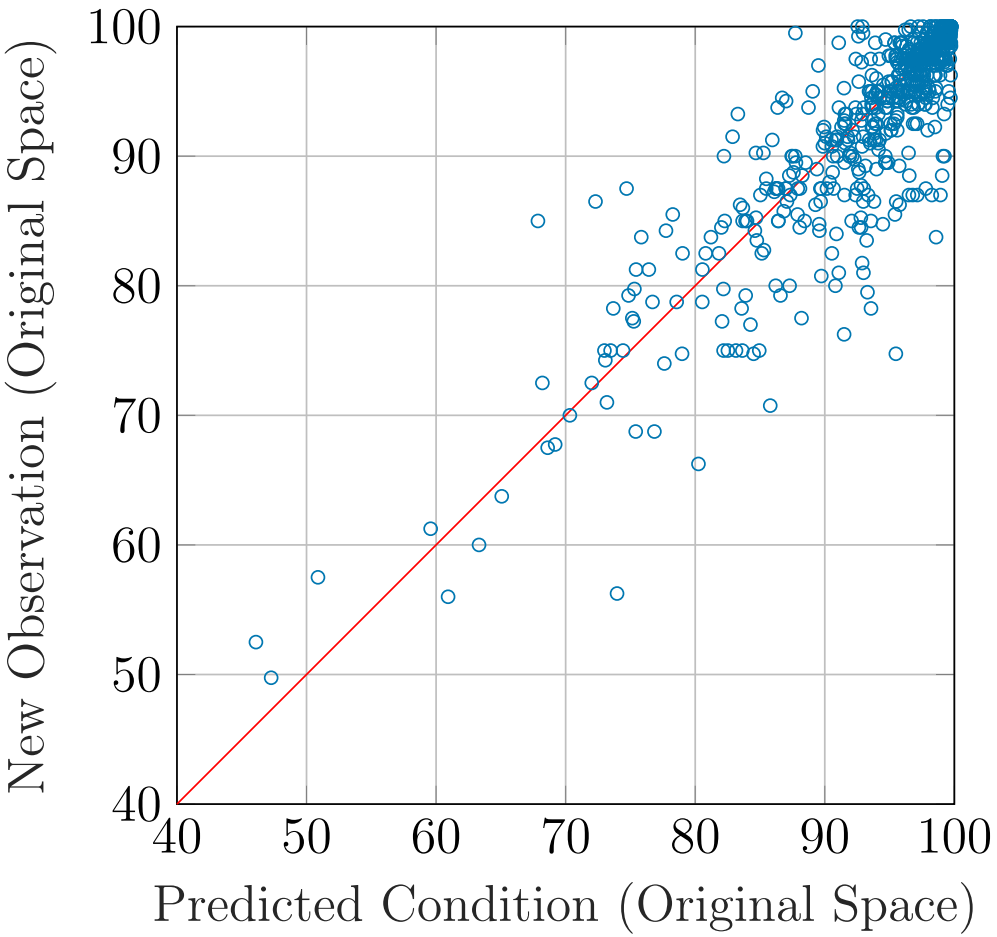
Real data

[Google images]

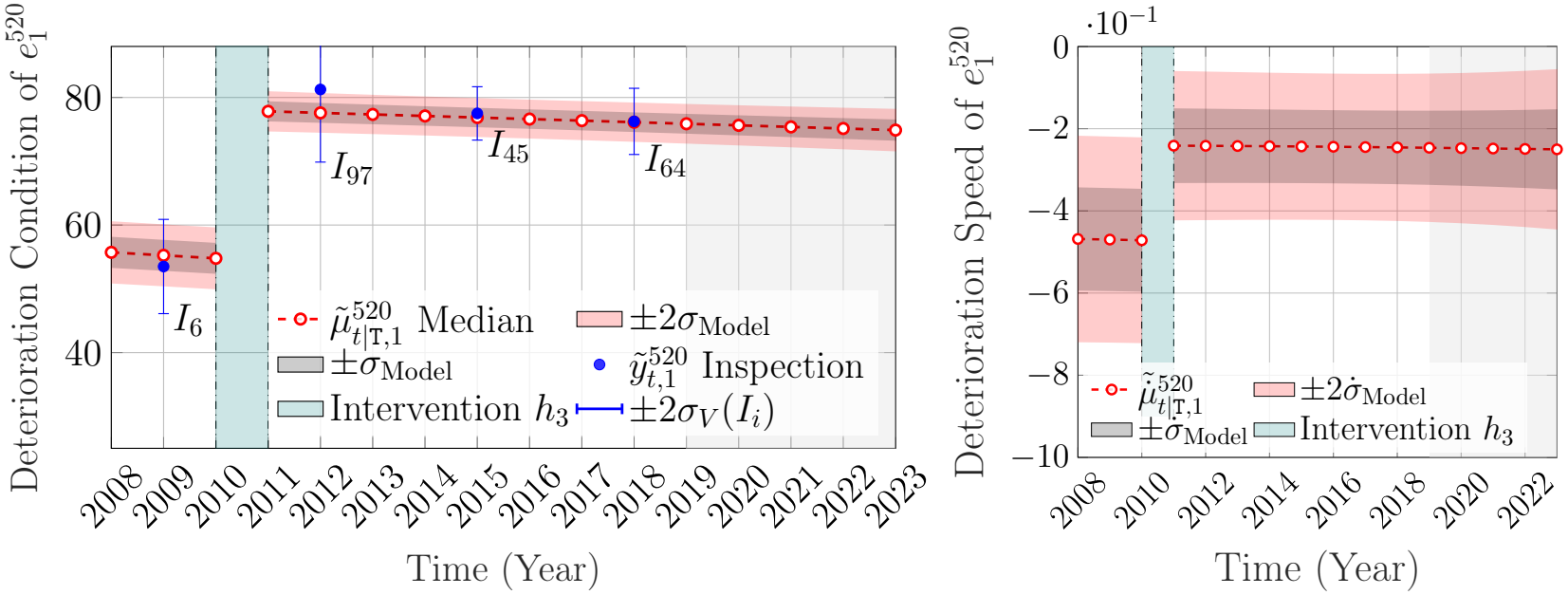
Example – Real data



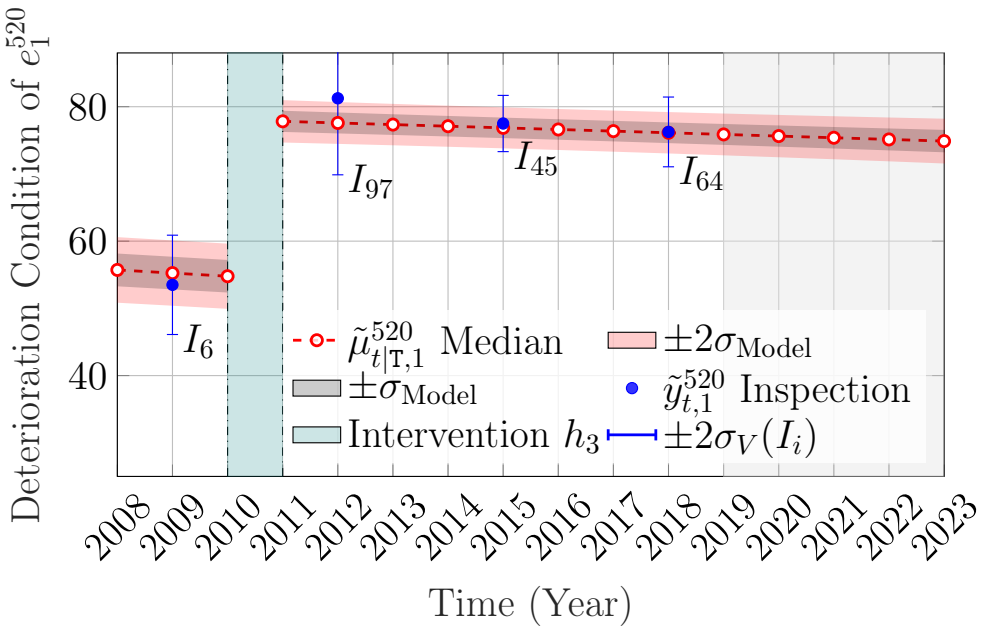
Validation set – Real data

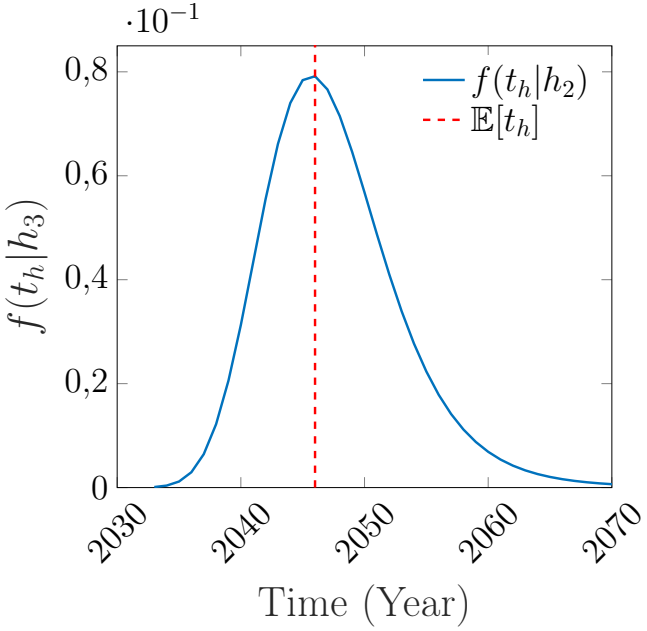


Quantifying the effect of interventions



Quantifying the effect of interventions

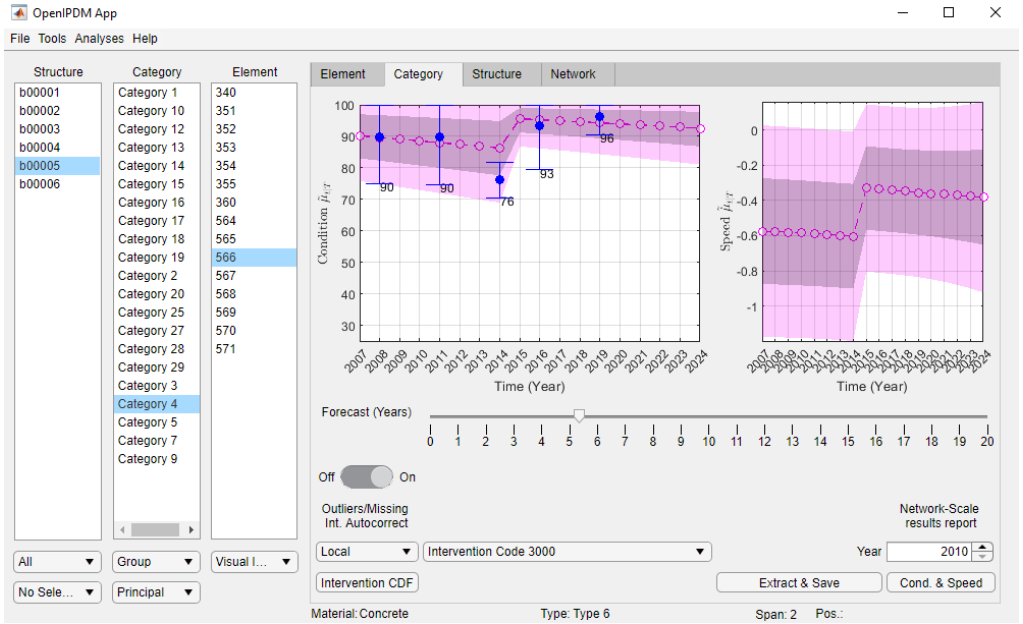




OpenIPDM

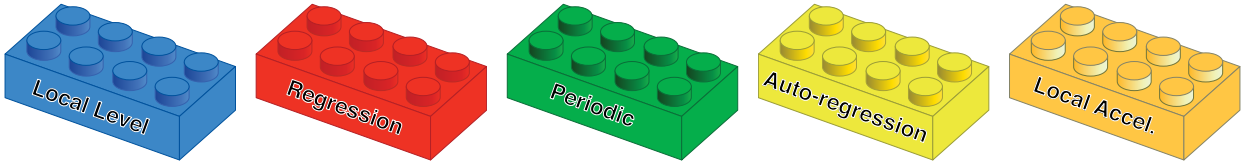
OpenBDLM is available on **GitHub** → Zachary Hamida

Tutorials are available on the YouTube channel



Summary

Bayesian Dynamic Linear Model (BDLM):



Transition model

$$\mathbf{x}_t = \mathbf{A}\mathbf{x}_{t-1} + \underbrace{\mathbf{w}_t}_{W \sim \mathcal{N}(\mathbf{0},\mathbf{Q})}$$

Observation model

$$\mathbf{y}_t = \mathbf{C}\mathbf{x}_t + \underbrace{\mathbf{v}_t}_{W \sim \mathcal{N}(\mathbf{0},\mathbf{R})}$$