# Probability distributions

(CIV6540 - Probabilistic Machine Learning for Civil Engineers)

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# Module #2 Outline

**Normal – $\mathcal{N}(\mu, \sigma^2)$**
**Lognormal –**
ln $\mathcal{N}(\lambda, \zeta)$
**Beta – $\mathcal{B}(\alpha, \beta)$**

## Topics organization

| | | |
|---|---|---|
| *Background* | 1 | Revision probability & linear algebra |
| | 2 | Probability distributions |
| *Machine Learning Basics* | 0 | Introduction |
| | 3 | Bayesian Estimation $p(A\vert B) = \frac{p(B\vert A)p(A)}{p(B)}$ |
| | 4 | MCMC sampling & Newton |
| *Supervised learning* | 5 | Regression |
| | 6 | Classification |
| | 7 | LSTM networks for time series |
| *Unsupervised learning* | 7 | State-space model for time-series |
| *Decision Making & RL* | 8 | Decision Theory |
| | 9 | AI & Sequential decision problems |

# Section Outline

# Univariate Normal, $X \sim \mathcal{N}(x; \mu, \sigma^2)$ [.m]

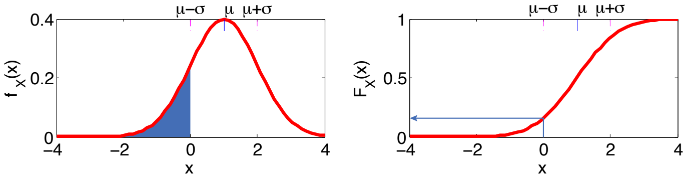
Probability density function (PDF) for a random variable $X : x \in \mathbb{R}$

$$f_X(x) = \mathcal{N}(x; \mu, \sigma^2) \begin{cases} \mu & : \text{mean (i.e, C.G.)} \\ \sigma^2 & : \text{variance (i.e, Inertia)} \end{cases}$$

Cumulative density function (CDF)

$$F_X(0) = \int_{-\infty}^{0} f_X(0)dx = 0.16$$

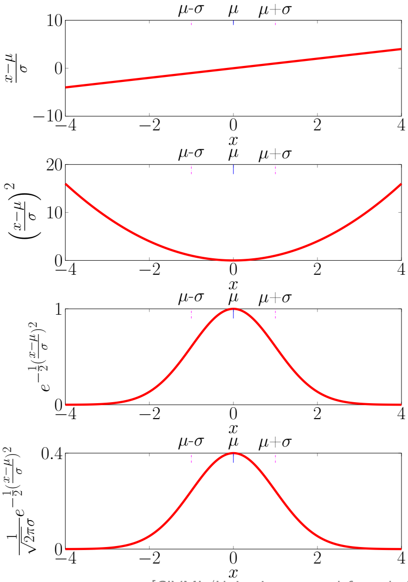
$\mu = 1$
$\sigma = 1$




[CIVML/Univariate_normal_example.m]

# Anatomy of $\mathcal{N}(x; \mu, \sigma^2)$ [.m]

$$f(x) = \overbrace{\frac{1}{\sqrt{2\pi}\sigma}}^{\text{norm. cte.}} \underbrace{\exp\left[-\frac{1}{2}\overbrace{\left(\underbrace{\frac{x-\mu}{\sigma}}_{\text{linear}}\right)^2}^{\text{quadratic}}\right]}_{\text{exponential}}$$



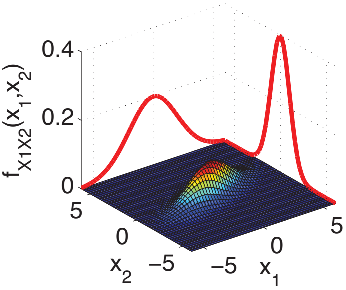

[CIVML/Univariate_normal_formulation.m]

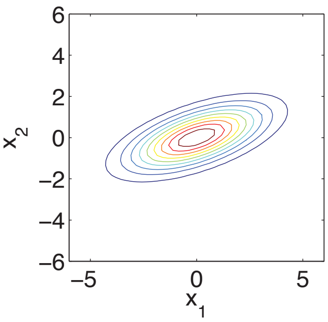
# Bivariate Normal [.m]

Probability density function (PDF) for 2 random variables $X_1, X_2$

$$f(x_1, x_2) = \frac{1}{2\pi\sigma_{X_1}\sigma_{X_2}\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x_1-\mu_{X_1}}{\sigma_{X_1}}\right)^2 + \left(\frac{x_2-\mu_{X_2}}{\sigma_{X_2}}\right)^2 - 2\rho\left(\frac{x_1-\mu_{X_1}}{\sigma_{X_1}}\right)\left(\frac{x_2-\mu_{X_2}}{\sigma_{X_2}}\right)\right]\right\}$$

$$\mu_{X_1} = 0$$
$$\sigma_{X_1} = 2$$
$$\mu_{X_2} = 0$$
$$\sigma_{X_2} = 1$$
$$\rho = 0.6$$



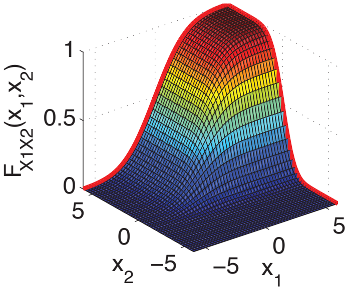

[CIVML/Multivariate_normal_example.m]

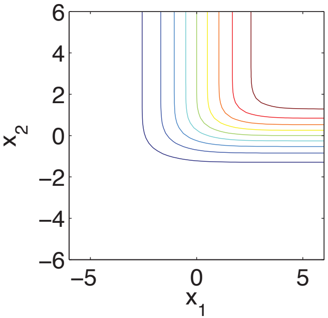
# Bivariate Normal [.m]

Cumulative distribution function (CDF) for 2 random variables $X_1, X_2$

$$F(x_1, x_2) = \int_{-\infty}^{x_1}\int_{-\infty}^{x_2} f(x_1, x_2)\partial x \partial y$$

$\mu_{X_1} = 0$
$\sigma_{X_1} = 2$
$\mu_{X_2} = 0$
$\sigma_{X_2} = 1$
$\rho = 0.6$




[CIVML/Multivariate_normal_example_CDF.m]

# Multivariate Normal, $\mathbf{X} \sim \mathcal{N}(\mathbf{x}; \mathbf{M_X}, \mathbf{\Sigma_X})$

Probability density function (PDF) for $n$ random variables $\mathbf{X} = [X_1, X_2, \cdots, X_n]^\intercal$, where $\mathbf{M_X} = [\mu_1, \mu_2, \cdots, \mu_n]^\intercal$ is a vector containing mean values and $\mathbf{\Sigma_X}$ is the covariance matrix.

$$\mathbf{\Sigma_X} = \begin{bmatrix} \sigma_{X_1}^2 & \rho_{12}\sigma_{X_1}\sigma_{X_2} & \cdots & \rho_{1n}\sigma_{X_1}\sigma_{X_n} \\ & \sigma_{X_2}^2 & \cdots & \rho_{2n}\sigma_{X_2}\sigma_{X_n} \\ & & \ddots & \vdots \\ \text{sym.} & & & \sigma_{X_n}^2 \end{bmatrix}_{n\times n}$$

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{1}{(2\pi)^{n/2}(\det \mathbf{\Sigma_X})^{1/2}} \exp\left[-\frac{1}{2}(\mathbf{x} - \mathbf{M_X})^\intercal \mathbf{\Sigma_X}^{-1}(\mathbf{x} - \mathbf{M_X})\right]$$

# Standard deviation, correlation and covariance matrices

- $\mathbf{D_X}$: Standard deviation matrix
- $\mathbf{R_X}$: Correlation matrix
- $\mathbf{\Sigma_X}$: Covariance matrix

$$\mathbf{D_X} = \begin{bmatrix} \sigma_{X_1} & 0 & 0 & 0 \\ & \sigma_{X_2} & 0 & 0 \\ & & \ddots & 0 \\ \text{sym.} & & & \sigma_{X_n} \end{bmatrix}, \ \mathbf{R_X} = \begin{bmatrix} 1 & \rho_{12} & \cdots & \rho_{1n} \\ & 1 & \cdots & \rho_{2n} \\ & & \vdots & \rho_{n-1n} \\ \text{sym.} & & & 1 \end{bmatrix}$$

$$\mathbf{\Sigma_X} = \mathbf{D_X R_X D_X} = \begin{bmatrix} \sigma^2_{X_1} & \rho_{12}\sigma_{X_1}\sigma_{X_2} & \cdots & \rho_{1n}\sigma_{X_1}\sigma_{X_n} \\ & \sigma^2_{X_2} & \cdots & \rho_{2n}\sigma_{X_2}\sigma_{X_n} \\ & & \vdots & \vdots \\ \text{sym.} & & & \sigma^2_{X_n} \end{bmatrix}$$

# Multivariate Normal – Properties

1. Completely defined by $\mathbf{M_X}$ and $\mathbf{\Sigma_X}$
2. Marginal distributions are also Normal

$$x_i : X_i \sim \mathcal{N}(x_i; [\mathbf{M_X}]_i, [\mathbf{\Sigma_X}]_{ii})$$

3. An absence of correlation implies independence

$$\rho_{ij} = 0 \Leftrightarrow X_i \perp\!\!\!\perp X_j$$

4. The asymptotical distribution obtained from the sum of (iid) random phenomenon is Normal (**Central Limit Theorem**)

Given, $X_i, \; i = 1, \cdots, n$ a set of independent identically distributed (iid) random variables
$Y = (X_1 + \cdots + X_n) \sim \mathcal{N}(y; \mu_Y, \sigma_Y^2)$, for $n \to \infty$

# Multivariate Normal – Properties (cont.)

5. Linear functions of Normal random variables are also Normal

$$\text{Given, } \mathbf{X} \sim \mathcal{N}(\mathbf{M}_X, \mathbf{\Sigma}_X) \text{ et } \mathbf{Y} = \mathbf{A}\mathbf{X} + \mathbf{b}$$

$$\mathbf{Y} \sim \mathcal{N}(\mathbf{A}\mathbf{M}_X + \mathbf{b}, \mathbf{A}\mathbf{\Sigma}_X\mathbf{A}^\intercal)$$

6. Conditional distributions are Normal

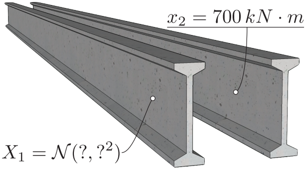
$$\mathbf{X} = \left\{ \begin{array}{c} \mathbf{X}_1 \\ \mathbf{X}_2 \end{array} \right\}, \ \mathbf{M} = \left\{ \begin{array}{c} \mathbf{M}_1 \\ \mathbf{M}_2 \end{array} \right\}, \ \mathbf{\Sigma} = \left[ \begin{array}{cc} \mathbf{\Sigma}_1 & \mathbf{\Sigma}_{12} \\ \mathbf{\Sigma}_{21} & \mathbf{\Sigma}_2 \end{array} \right]$$

$$f_{\mathbf{X}_1|\mathbf{X}_2}(\mathbf{x}_1 | \underbrace{\mathbf{X}_2 = \mathbf{x}_2}_{\text{observations}}) = \frac{f_{\mathbf{X}_1\mathbf{X}_2}(\mathbf{x}_1, \mathbf{x}_2)}{f_{\mathbf{X}_2}(\mathbf{x}_2)} = \mathcal{N}(\mathbf{x}_1; \mathbf{M}_{1|2}, \mathbf{\Sigma}_{1|2})$$

$$\mathbf{M}_{1|2} = \mathbf{M}_1 + \mathbf{\Sigma}_{12}\mathbf{\Sigma}_2^{-1}(\mathbf{x}_2 - \mathbf{M}_2), \quad \mathbf{\Sigma}_{1|2} = \mathbf{\Sigma}_1 - \mathbf{\Sigma}_{12}\mathbf{\Sigma}_2^{-1}\mathbf{\Sigma}_{21}$$

$$\text{2D: } \mu_{1|2} = \mu_1 + \rho\sigma_1 \frac{x_2 - \mu_2}{\sigma_2}, \quad \sigma_{1|2} = \sigma_1\sqrt{1 - \rho^2}$$

# Example – Multivariate Normal conditional distributions




The prior knowledge of the resistance $(X_1, X_2)$ for two adjacent beams is

$$\begin{array}{lcll} X_1 & \sim & \mathcal{N}(x_1; 500, 150^2) & [\text{kN}\cdot\text{m}] \\ X_2 & \sim & \mathcal{N}(x_2; 500, 150^2) & [\text{kN}\cdot\text{m}] \end{array}$$

Knowing that the resistance of beams is correlated: $\rho_{12} = 0.8$, **what is the strength $X_1$ given that we have observed that $x_2 = 700\,\text{kN}\cdot\text{m}$?**

$$f_{X_1|X_2}(x_1|x_2) = \mathcal{N}(x_1; \mu_{1|2}, \sigma^2_{1|2}) = \mathcal{N}(x_1; 660, 90^2)$$

[Radio-Canada]

# Example – Multivariate Normal conditional (cont.)

$$f_{X_1 X_2}(x_1, x_2) = \mathcal{N}(\mathbf{M_X}, \mathbf{\Sigma_X}) \begin{cases} \mathbf{M_X} = \begin{bmatrix} 500 \\ 500 \end{bmatrix} \\ \mathbf{\Sigma_X} = \begin{bmatrix} 150^2 & 0.8 \cdot 150^2 \\ 0.8 \cdot 150^2 & 150^2 \end{bmatrix} \end{cases}$$

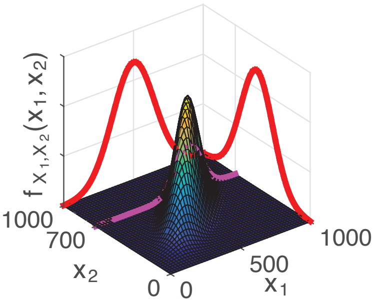
$$f_{X_1|X_2}(x_1|x_2) = \frac{f_{X_1 X_2}(x_1, x_2)}{f_{X_2}(x_2)} = \mathcal{N}(\mu_{1|2}, \sigma^2_{1|2}) \begin{cases} \mu_{1|2} = \mu_1 + \rho\sigma_1 \frac{x_2 - \mu_2}{\sigma_2} \\ \sigma_{1|2} = \sigma_1\sqrt{1 - \rho^2} \end{cases}$$

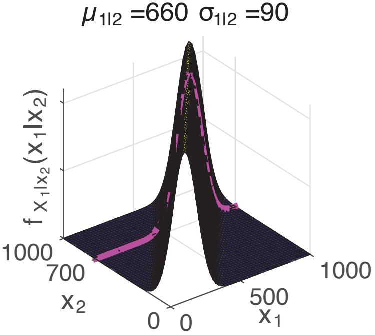
$$\mu_{1|2} = 500 + 0.8 \times 150 \frac{\overbrace{700}^{\text{observation}} - 500}{150} = 660$$

$$\sigma_{1|2} = 150\sqrt{1 - 0.8^2} = 90$$

# Example – Multivariate Normal conditional [.m]

$$f_{X_1|X_2}(x_1|x_2) = \frac{f_{X_1X_2}(x_1, x_2)}{f_{X_2}(x_2)} = \mathcal{N}(\mu_{1|2}, \sigma^2_{1|2}) \begin{cases} \mu_{1|2} = \mu_1 + \rho\sigma_1 \frac{x_2 - \mu_2}{\sigma_2} \\ \sigma_{1|2} = \sigma_1\sqrt{1-\rho^2} \end{cases}$$




[CIVML/poutres_conditionnelles.m]

# Sum of Normal random variables

Given 2 random variables $X \sim \mathcal{N}(x; \mu_X, \sigma_X^2)$, $Y \sim \mathcal{N}(y; \mu_Y, \sigma_Y^2)$

**If $X \perp\!\!\!\perp Y$**

$$\begin{aligned} Z &= X + Y \\ &\sim \mathcal{N}(z; \underbrace{\mu_X + \mu_Y}_{\mu_Z}, \underbrace{\sigma_X^2 + \sigma_Y^2}_{\sigma_Z^2}) \end{aligned}$$

**If $\rho_{XY} \neq 0$**

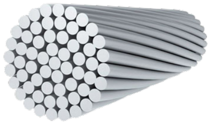
$$\begin{aligned} Z &= X + Y \\ &\sim \mathcal{N}(z; \underbrace{\mu_X + \mu_Y}_{\mu_Z}, \underbrace{\sigma_X^2 + \sigma_Y^2 + 2\rho_{XY}\sigma_X\sigma_Y}_{\sigma_Z^2}) \end{aligned}$$

$$\begin{aligned} Z &= \textstyle\sum_{i=1}^{n} X_i \\ &\sim \mathcal{N}\left(z; \textstyle\sum_{i=1}^{n} \mu_{X_i}, \sum_{i=1}^{n}\sum_{j=1}^{n} \text{cov}(X_i, X_j)\right) \end{aligned}$$

(Note: $\text{cov}(X_i, X_j) = [\boldsymbol{\Sigma}]_{ij}$)

# Example – Sum of Normal random variables

Given a cable made of 50 steel wires (ductile failure) each having a resistance $X_i \sim \mathcal{N}(x_i; 10, 3^2)\, kN$.



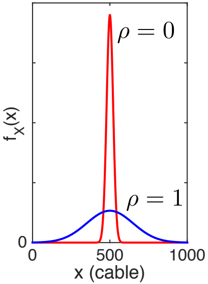
**What is $X_{\text{cable}} = \sum_{i=1}^{50} X_i$?**

With the hypothesis $X_i \perp\!\!\!\perp X_j$

$$X_{\text{cable}} \sim \mathcal{N}(x_{\text{cable}}; 50 \times 10, \underbrace{50 \times 3^2}_{\sigma_{X_{\text{cable}}}=3\sqrt{50}\approx 21\, kN})$$

With the hypothesis $\rho_{X_i,X_j} = 1$

$$X_{\text{cable}} \sim \mathcal{N}(x_{\text{cable}}; 50 \times 10, \underbrace{50^2 \times 3^2}_{\sigma_{X_{\text{cable}}}=3\times 50=150\, kN})$$




[steelwirerope.com, Der Kiureghian (2005)]

# : $X \sim \mathcal{N}(\mu, \sigma^2)$ [ ]

```
from scipy.stats import norm
from matplotlib import pyplot as plt, cm
import numpy as np

m_X = 0                         # Mean of X
s_X = 1                         # Standard deviation of X
x = np.arange(-3, 3, 0.01)     # Values of x to be evaluated
f_x = norm.pdf(x,m_X,s_X)      # PDF of X
F_x = norm.cdf(x,m_X,s_X)      # CDF of X
plt.plot(x, f_x)               # Plot the PDF of X
plt.show()
```

# : $\mathbf{X} \sim \mathcal{N}(\mathbf{M_X}, \mathbf{\Sigma_{XX}})$ []

```
from scipy.stats import multivariate_normal

m_X1 = 0              # Mean X1
m_X2 = 0              # Mean X2
s_X1 = 2              # Standard deviation X1
s_X2 = 1              # Standard deviation X2
rho = 0.6             # Correlation coefficient
x1x2 = [-2,2]         # Values of x to be evaluated

M_X = [m_X1,m_X2]     #Mean vector
S_XX = [[s_X1**2, rho*s_X1*s_X2],[rho*s_X1*s_X2, s_X2**2]]
f_x1x2 = multivariate_normal.pdf(x1x2,M_X,S_XX) #Joint PDF
F_x1x2 = multivariate_normal.cdf(x1x2,M_X,S_XX) #Joint CDF
```

## : $\mathbf{X} \sim \mathcal{N}(\mathbf{M_X}, \mathbf{\Sigma_{XX}})$ [ ]

```
from scipy.stats import multivariate_normal
from matplotlib import pyplot as plt, cm
import numpy as np

m_X1, m_X2 = 0, 0 # Mean X1/ Mean X2
s_X1, s_X2 = 2, 1 # Standard deviation X1 / Standard deviation X2
rho = 0.6         # Correlation coefficient
M_X = [m_X1, m_X2]
S_XX = [[s_X1**2, rho*s_X1*s_X2],[rho*s_X1*s_X2, s_X2**2]]
ndiv, min_x, max_x = 50, -5, 5 # Number of subdivision for plotting / min / max
x1 = np.linspace(min_x,max_x,ndiv)
x2 = np.linspace(min_x,max_x,ndiv)
x1M,x2M = np.meshgrid(x1,x2)          # Grid of x1,x2 to be evaluated
x1MR = np.reshape(x1M,(ndiv**2,1))    # Reshape grids in columns vectors
x2MR = np.reshape(x2M,(ndiv**2,1))
xMR=np.column_stack((x1MR,x2MR))      # Concatenate x1,x2 vectors
f_x1x2MR=multivariate_normal.pdf(xMR,M_X,S_XX)       # Evaluate each grid point
f_x1x2=np.reshape(f_x1x2MR,(ndiv,ndiv)).transpose()  # Reshape: vector to grid
fig = plt.figure()
ax = fig.add_subplot(2, 1, 1, projection='3d')  # First of 2 plots on one fig.
ax.plot_surface(x1M,x2M,f_x1x2,rstride=1, cstride=1, cmap=cm.coolwarm)
ax.set(xlabel='$x_1$', ylabel='$x_2$', zlabel = '$f_{X1X2}(x_1,x_2)$')
ax = fig.add_subplot(2, 1, 2)
ax.contour(x1, x2, f_x1x2)                     # 2D contour plot
ax.set_xlim([min_x,max_x])
ax.set_ylim([min_x,max_x])
ax.set(xlabel='$x_1$', ylabel='$x_2$')
plt.show()
```

# Section Outline

# Univariate Lognormal

The random variable $X$ **is Lognormal if** $\ln X$ **is Normal**.

The univariate Lognormal PDF is defined from the mean $(\mu_{\ln X} = \lambda)$ and variance $(\sigma^2_{\ln X} = \zeta^2)$ defined in the Lognormal space $(\ln X)$.

- $\mu_X$: mean of $X$
- $\lambda$: mean of $\ln X$ $(= \mu_{\ln X})$

- $\sigma^2_X$: variance of $X$
- $\zeta^2$: variance of $\ln X$ $(= \sigma^2_{\ln X})$

$$\lambda = \mu_{\ln X} = \ln \mu_X - \frac{\zeta^2}{2}$$

$$\zeta = \sigma_{\ln X} = \sqrt{\ln(1 + \delta_X^2)}$$

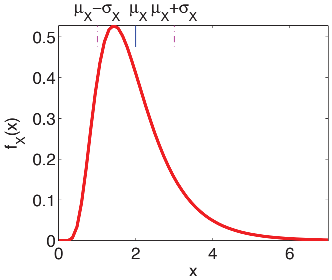
$$(\zeta \cong \delta_X \text{ for } \delta_X < 0.3)$$

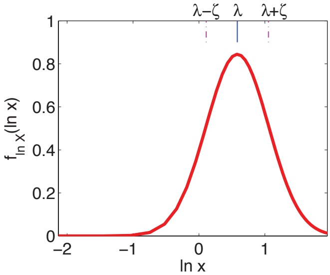
# Univariate Lognormal, $X \sim \ln\mathcal{N}(\lambda, \zeta)$ [.m]

The probability density function (PDF) for $X : x \in \mathbb{R}^+$

$$f_X(x) = \frac{1}{x\sqrt{2\pi}\zeta} \exp\left[-\frac{1}{2}\left(\frac{\ln x - \lambda}{\zeta}\right)^2\right], \quad x > 0$$

$\mu_X = 2$
$\sigma_X = 1$
$\lambda = 0.58$
$\zeta = 0.47$




[CIVML/Univariate_lognormal_example.m]

# Formulation – Univariate Lognormal, $X \sim \ln\mathcal{N}(x; \lambda, \zeta)$

$$f_X(x) = \frac{1}{x\sqrt{2\pi}\zeta} \exp\left[-\frac{1}{2}\left(\frac{\ln x - \lambda}{\zeta}\right)^2\right], \quad x > 0$$

**How do we obtain the Lognormal PDF?**

# Formulation – Univariate Lognormal, $X \sim \ln\mathcal{N}(x; \lambda, \zeta)$

$$\left.\begin{array}{rcl} X & : & x \in \mathbb{R}^+ \\ X' = \ln X & : & x' \in \mathbb{R} \end{array}\right\}$$

$$\begin{array}{rcl} \overbrace{f_{X'}(x')}^{\mathcal{N}(x';\lambda,\zeta^2)}\, dx' & = & f_X(x)dx \\ f_{X'}(x')\left|\frac{dx'}{dx}\right| & = & f_X(x) \end{array}$$

$$\frac{dx'}{dx} = \frac{d\ln x}{dx} = \frac{1}{x}$$

$$\begin{array}{rcl} f_X(x) & = & \frac{1}{x} \cdot \mathcal{N}(\ln x; \lambda, \zeta^2) \\ & = & \frac{1}{x} \cdot \frac{1}{\sqrt{2\pi}\zeta} \exp\left[-\frac{1}{2}\left(\frac{\ln x - \lambda}{\zeta}\right)^2\right], \quad x > 0 \end{array}$$

# Multivariate Lognormal

$X_1, X_2, \cdots, X_n$ are jointly Lognormal is $\ln X_1, \ln X_2, \cdots, \ln X_n$ are jointly Normal.

The multivariate Lognormal PDF is defined form the mean values ($\mu_{\ln X_i} = \lambda$), variances ($\sigma^2_{\ln X_i} = \zeta^2$) and correlation coefficients ($\rho_{\ln X_i \ln X_j}$) for $\ln X_i,\ i = 1, \cdots, n$ (**in the log space**).

$$\rho_{\ln X_i \ln X_j} = \frac{1}{\zeta_i \zeta_j} \ln(1+\rho_{X_i X_j} \delta_{X_i} \delta_{X_j}), \quad (\rho_{\ln X_i \ln X_j} \cong \rho_{X_i X_j} \text{ for } \delta_{X_i} \ll 0.3)$$

# Bivariate Lognormal [.m]

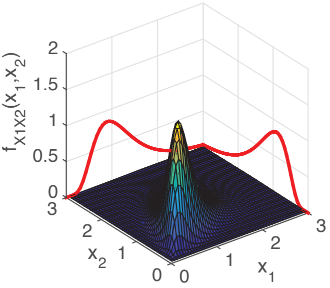
The PDF for 2 random variables $X_1, X_2$

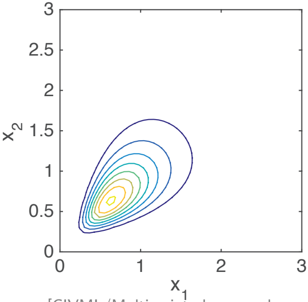
$$f_{X_1,X_2}(x_1,x_2) =$$

$$\frac{1}{x_1 x_2 \sqrt{2\pi}\zeta_{X_1}\zeta_{X_2}\sqrt{1-\rho_{\ln}^2}} \exp\left\{-\frac{1}{2(1-\rho_{\ln}^2)}\left[\left(\frac{\ln x_1 - \lambda_{X_1}}{\zeta_{X_1}}\right)^2 + \left(\frac{\ln x_2 - \lambda_{X_2}}{\zeta_{X_2}}\right)^2 - 2\rho_{\ln}\left(\frac{\ln x_1 - \lambda_{X_1}}{\zeta_{X_1}}\right)\left(\frac{\ln x_2 - \lambda_{X_2}}{\zeta_{X_2}}\right)^2\right]\right\}, \quad \begin{cases} x_1, x_2 > 0 \\ \rho_{\ln} = \rho_{\ln X_1 \ln X_2} \end{cases}$$

$\mu_{X_i} = 1$
$\sigma_{X_i} = 0.5$
$\rho_{X_1 X_2} = 0.6$

$\lambda_i = -0.11$
$\zeta_i = 0.47$
$\rho_{ln} = 0.62$




[CIVML/Multivariate_lognormal_example.m]

# Multivariate Lognormal – Properties

1. Completely defined by $\mathbf{M_X}$ et $\mathbf{\Sigma_X}$
2. Marginal distributions are also Lognormal
3. Conditional distributions are Lognormal
$$x_i : X_i \sim \ln\mathcal{N}(x_i; [\mathbf{M}_{\ln \mathbf{x}}]_i, [\mathbf{\Sigma}_{\ln \mathbf{x}}]_{ii})$$
4. The absence of correlation imply independence
$$\rho_{ij} = 0 \Leftrightarrow X_i \perp\!\!\!\perp X_j$$
5. The multiplication of joint Lognormal random variables is jointly Lognormal.

$$X \sim \ln\mathcal{N}(x; \lambda_X, \zeta_X),\ \ Y \sim \ln\mathcal{N}(y; \lambda_Y, \zeta_Y)$$

$$\left.\begin{array}{rcl} Z & = & X \cdot Y \\ & \sim & \ln\mathcal{N}(z; \lambda_Z, \zeta_Z) \end{array}\right\} \begin{array}{l} \lambda_Z = \lambda_X + \lambda_Y \\ \zeta_Z^2 = \zeta_X^2 + \zeta_Y^2 \end{array}$$

# Multivariate Lognormal – properties (cont.)

6. The asymptotic distribution obtained from the product of iid Lognormal random variables is Lognormal (Central Limit theorem)

   Given, $X_i, \;\; i = 1, \cdots, n$ a set of independent identically distributed (iid) random variables

$$\begin{aligned} Y &= X_1 \cdot X_2 \cdot \ldots \cdot X_n \\ &\sim \ln\mathcal{N}(y; \mu_Y, \sigma_Y^2), \text{ for } n \to \infty \end{aligned}$$

# : $X \sim \ln\mathcal{N}(\lambda, \zeta)$ []

```
from scipy.stats import lognorm, norm
from sympy import Symbol
import sympy as sp
import numpy as np

m_X = 1      # Mean of X
s_X = 1      # Standard deviation of X
s_lnX = np.sqrt(np.log(1+(s_X/m_X)**2)) # Std log(X)
m_lnX = np.log(m_X)-0.5*s_lnX**2;       # Mean log(X)

x = np.arange(-3, 3, 0.01) # Values of x to be evaluated

f_x=lognorm.pdf(x,s_lnX, scale=np.exp(m_lnX)) # PDF of X
f_x=norm.pdf(np.log(x),m_lnX ,s_lnX)/x         # PDF of X

# Analytic formulation
x_ = Symbol('x_', positive=True) # Define a positive analytic variable
f_x=1/(sp.sqrt(2*np.pi)*s_lnX*x_)*sp.exp(-1/2*((sp.log(x_)-m_lnX)/s_lnX)**2)#PDF
F_x=sp.integrate(f_x,(x_,0,x_))  # CDF
```

# : $X \sim \ln\mathcal{N}(\lambda, \zeta)$ [ ]

```
from scipy.stats import multivariate_normal
import numpy as np

m_X1  = 1               # Mean X1
m_X2  = 1               # Mean X2
s_X1  = 0.5             # Standard deviation X1
s_X2  = 0.5             # Standard deviation X2
rho  = 0.6              # Correlation coefficient
x1x2 = [2,2]            # Values of x to be evaluated

s_ln  = lambda m,s : np.sqrt(np.log(1+(s/m)**2)) # std
m_ln  = lambda m,s_ln_ : np.log(m)-0.5*s_ln_**2  # mean

s_lnX1  = s_ln(m_X1,s_X1)                        # std ln(X1)
m_lnX1  = m_ln(m_X1,s_lnX1)                      # mean ln(X1)
s_lnX2  = s_ln(m_X2,s_X2)                        # std ln(X2)
m_lnX2  = m_ln(m_X2,s_lnX2)                      # mean ln(X2)

rho_lnX1X2=1/(s_lnX1*s_lnX2)*np.log(1+rho*(s_X1/m_X1)*(s_X2/m_X2))# Corr. coeff.

M_lnX  = [m_lnX1,m_lnX2]
S_lnXX  = [[s_lnX1**2, rho_lnX1X2*s_lnX1*s_lnX2],\
            [rho_lnX1X2*s_lnX1*s_lnX2, s_lnX2**2]]
f_x1x2MR=multivariate_normal.pdf(np.log(x1x2),M_lnX,S_lnXX)/np.prod(x1x2,axis=0)
```

# : $\mathbf{X} \sim \ln\mathcal{N}(\mathbf{x}; \boldsymbol{\lambda}, \boldsymbol{\zeta})$ [.m]

```
%%Code snippet - Matlab multivariate log normal R.V.
m_X1=1;     % Mean X1
m_X2=1;     % Mean X2
s_X1=0.5;   % Standard deviation X1
s_X2=0.5;   % Standard deviation X2
rho=0.6;    % Corr. coeff.
x1x2=[-2,2];% Values of x to be evaluated

s_ln=@(m,s) sqrt(log(1+(s/m)^2)); % std
m_ln=@(m,s_ln) log(m)-0.5*s_ln^2; % mean

s_lnX1=s_ln(m_X1,s_X1);           % std ln(X1)
m_lnX1=m_ln(m_X1,s_lnX1);         % mean ln(X1)
s_lnX2=s_ln(m_X2,s_X2);           % std ln(X2)
m_lnX2=m_ln(m_X2,s_lnX2);         % mean ln(X2)

rho_lnX1X2=1/(s_lnX1*s_lnX2)*log(1+rho*(s_X1/m_X1)*(s_X2/m_X2));  % Corr. coeff.

M_lnX=[m_lnX1,m_lnX2];
S_lnX=[s_lnX1^2 rho_lnX1X2*s_lnX1*s_lnX2;rho_lnX1X2*s_lnX1*s_lnX2 s_lnX2^2];
f_x1x2MR=mvnpdf(log(x1x2),M_lnX,S_lnX)./prod(x1x2,2);
```

# Section Outline

# Beta, $X \sim \mathcal{B}(x; \alpha, \beta)$ [.m]

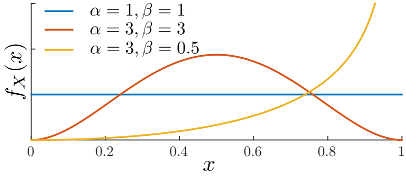
Probability density function (PDF) for $X : x \in (0, 1)$

$$f(x) = \frac{x^{\alpha-1}(1-x)^{\beta-1}}{\mathrm{B}(\alpha, \beta)} \begin{cases} \alpha > 0 \\ \beta > 0 \\ \mathrm{B}(\alpha, \beta): \text{ Fonction Beta} \end{cases}$$

Given two events ME&CE

$$\mathcal{S} = \{A, \overline{A}\} \begin{cases} \Pr(A) = X \\ \Pr(\overline{A}) = 1 - X \end{cases}$$

$\alpha$ : # of observation of $A$
$\beta$ : # of observation of $\overline{A}$



# $\mathrm{B}(\alpha, \beta)$: Beta function

Probability density function (PDF) for $X : x \in (0, 1)$

$$\left.\begin{aligned} f(x) &= \frac{x^{\alpha-1}(1-x)^{\beta-1}}{\mathrm{B}(\alpha,\beta)} \end{aligned}\right\} \mathrm{B}(\alpha,\beta): \text{ integration cte.}$$

$$\propto x^{\alpha-1}(1-x)^{\beta-1}$$

$$\mathrm{B}(\alpha,\beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}dx$$

# Example – Beta PDF [.m]

$$\mathcal{S} = \{\text{Head}, \text{Tail}\}, \quad \Pr(\{\text{Head}\}) = X$$

[CIVML/Beta_example_2.m]

# : $X \sim \text{Beta}(\alpha, \beta)$ []

```
from scipy.stats import beta
from matplotlib import pyplot as plt, cm
import numpy as np

a = 3 #\alpha
b = 2 #\beta

x = np.arange(0, 1, 0.001) # Values of x to be evaluated
f_x = beta.pdf(x,a,b)      # PDF of X

plt.plot(x, f_x)           # Plot the PDF of X
plt.xlabel('$x$')
plt.ylabel('$f_X(x)$')
plt.show()
```

# Summary

**Univariate Normal**:
$X \sim \mathcal{N}(x; \mu, \sigma^2), x \in (-\infty, +\infty)$

if $X \sim \mathcal{N}(x; \mu_X, \sigma_X^2),\ \ Y \sim \mathcal{N}(y; \mu_Y, \sigma_Y^2)$

$$\begin{aligned} Z &= X + Y \\ &\sim \mathcal{N}(z; \mu_Z, \sigma_Z^2) \end{aligned}$$

**Multivariate Normal**:
$\mathbf{X} \sim \mathcal{N}(x; \mathbf{M_X}, \mathbf{\Sigma_X})$

**Normal conditional**:
$f_{\mathbf{X}_1|\mathbf{X}_2}(\mathbf{x}_1|\mathbf{X}_2 = \mathbf{x}_2) = \mathcal{N}(\mathbf{x}_1; \mathbf{M}_{1|2}, \mathbf{\Sigma}_{1|2})$

**Univariate Lognormal**:
$X \sim \ln\mathcal{N}(x; \lambda, \zeta), x \in (0, +\infty)$

if
$X \sim \ln\mathcal{N}(x; \lambda_X, \zeta_X^2),\ \ Y \sim \ln\mathcal{N}(y; \lambda_Y, \zeta_Y^2)$

$$\begin{aligned} Z &= X \cdot Y \\ &\sim \ln\mathcal{N}(z; \lambda_Z, \zeta_Z^2) \end{aligned}$$

**Beta**:
$X \sim \text{Beta}(x; \alpha, \beta), x \in (0, 1)$