# Module #7
# Estimation de la fiabilité: Analyses de sensibilité

(CIV8530 - Fiabilité des structures et systèmes)

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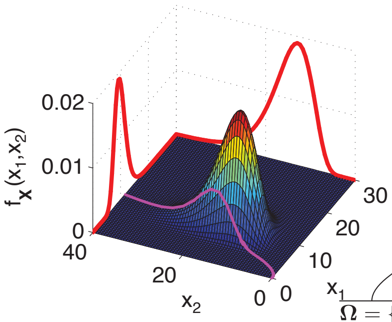
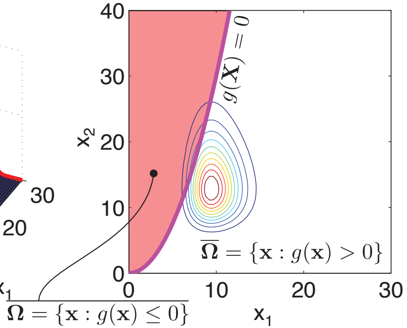
Section 14.3.3 - A. Der Kiureghian (2005). *First- and second-order reliability methods*. Chapter 14 in Engineering design reliability handbook, CRC Press

Section 8.5 - Nowak & Collins (2013). *Reliability of Structures.*, CRC Press

# Probabilité de défaillance - Analyses de sensibilité

| | |
|---|---|
| $\mathbf{X} = \{X_1, X_2, \cdots, X_N\}$: | ensemble de variables aléatoires |
| $f_{\mathbf{X}}(\mathbf{x})$: | densité de probabilité conjointe |
| $g(\mathbf{X})$: | fonction d'état limite |
| $g(\mathbf{X}) = 0$: | surface d'état limite |

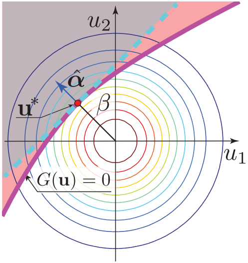


# Problématique?

on désire estimer

$$\boxed{\beta_{\text{nouveau}} = \beta_{\text{ancien}} + \sum_i \frac{\partial \beta}{\partial \theta_i} \Delta \theta_i}$$




1. Quelle est l'importance relative de $X_i$ dans $p_f$ et $\beta$? $\to$ **Calibration de codes**
2. Quelle est la sensibilité de $p_f$ et $\beta$ par rapport aux paramètres ($\boldsymbol{\theta}_f$) de $f_{\mathbf{X}}(\mathbf{x})$ ? Imp($\theta \to p_f$)
3. Quelle est la sensibilité de $p_f$ et $\beta$ par rapport aux valeurs moyennes et écarts types ($\mathbf{M_X}, \mathbf{D_X}$) de $\mathbf{X}$? Imp([$\mathbf{M_X}, \mathbf{D_X}$] $\to p_f$)

**Solution**: Approximations analytiques basées sur FORM

# Plan du module #7

**Introduction**

**$\beta-\alpha\mathbf{u}$**

**Imp($X_i \rightarrow Z$)**

**Calibration des codes**

**Imp($\theta \rightarrow p_f$)**

**Imp([$\mathbf{M_X}, \mathbf{D_X}$] $\rightarrow p_f$)**

## Organisation de la matière

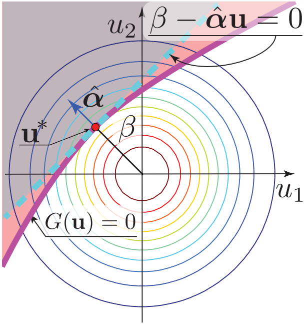
| | | |
|---|---|---|
| *Théorie probabilité* | 0 | Révision algèbre & probabilités |
| | 1 | Lois de probabilités |
| | 3 | PDF multivariés |
| *Fiabilité des structures & systèmes* | | Introduction |
| | 2 | Formulation fiabilité composantes |
| | 9 | Formulation fiabilité systèmes |
| *Estimation – $p_f$ échantillonnage* | 4 | Monte Carlo |
| | 12 | Échantillonnage par importance |
| *Estimation – $p_f$ analytique* | 5 | FOSM |
| | 6 | FORM |
| | 8 | SORM |
| *Utilisation des résultats* | 7 | Analyse de sensibilité |
| | 10 | Incertitudes aléatoires & épistémiques |
| | 11 | Données empiriques |
| | 13 | Prise de décision |
| | 14 | Métamodèles & modèles empiriques |

# Formulation alternative – Surface d'état limite

$$
\begin{aligned}
G(U) &\cong \overbrace{G(\mathbf{u}^*)}^{\equiv 0} + \nabla G(\mathbf{u}^*)(\mathbf{U} - \mathbf{u}^*) \\
&= \nabla G(\mathbf{u}^*)\mathbf{U} - \nabla G(\mathbf{u}^*)\mathbf{u}^* \\
&= \frac{\|\nabla G(\mathbf{u}^*)\|}{\|\nabla G(\mathbf{u}^*)\|}(\nabla G(\mathbf{u}^*)\mathbf{U} - \nabla G(\mathbf{u}^*)\mathbf{u}^*) \\
&= \|\nabla G(\mathbf{u}^*)\|(\beta - \hat{\boldsymbol{\alpha}}\mathbf{U})
\end{aligned}
$$

$$
\boxed{\beta \cong \frac{\mu_{G(U)}}{\sigma_{G(U)}} = \underbrace{-\frac{\nabla G(\mathbf{u}^*)}{\|\nabla G(\mathbf{u}^*)\|}}_{\hat{\boldsymbol{\alpha}}}\mathbf{u}^* = \hat{\boldsymbol{\alpha}}\mathbf{u}^*}
$$

$$p_f = \Pr(\beta - \hat{\boldsymbol{\alpha}}\mathbf{U} \leq 0)$$



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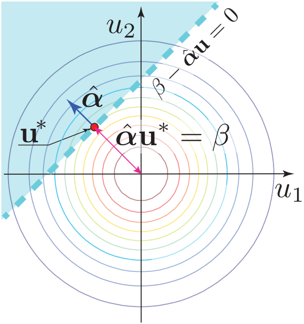


$$\boxed{\beta \cong \frac{\mu_{G(U)}}{\sigma_{G(U)}} = \underbrace{-\frac{\nabla G(\mathbf{u}^*)}{\|\nabla G(\mathbf{u}^*)\|}}_{\hat{\boldsymbol{\alpha}}}\mathbf{u}^* = \hat{\boldsymbol{\alpha}}\mathbf{u}^*}$$

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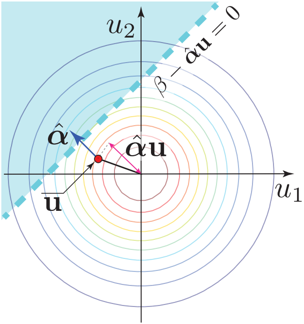


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# Formulation alternative – Surface d'état limite

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\end{aligned}
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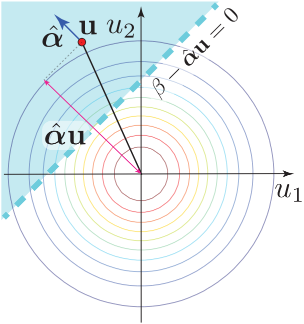
$$p_f = \Pr(\beta - \hat{\boldsymbol{\alpha}}\mathbf{U} \leq 0)$$




$$\boxed{\beta \cong \frac{\mu_{G(U)}}{\sigma_{G(U)}} = \underbrace{-\frac{\nabla G(\mathbf{u}^*)}{\|\nabla G(\mathbf{u}^*)\|}}_{\hat{\boldsymbol{\alpha}}}\mathbf{u}^* = \hat{\boldsymbol{\alpha}}\mathbf{u}^*}$$

# Plan de la section

# Rappel – marge de sécurité

Soit $R$ la *capacité* et $S$ la *demande*. La *marge de sécurité* (Z) est la distance entre $R$ et $S$

$$Z = R - S$$

où $\{Z \leq 0\}$ dénote la défaillance. Donc,

$$p_f = \Pr(Z \leq 0)$$

# $X_i$ indépendants

Soit la fonction d'état limite

$$Z = g(\mathbf{X}) = G(\mathbf{U})$$

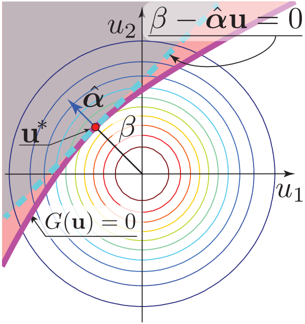
Après avoir linéarisé

$$Z \cong \|\nabla G(\mathbf{u}^*)\|(\beta - \hat{\boldsymbol{\alpha}}\mathbf{U})$$

où également

$$\frac{Z}{\|\nabla G(\mathbf{u}^*)\|} \cong \beta - \hat{\boldsymbol{\alpha}}\mathbf{U} = \beta - (\alpha_1 U_1 + \ldots + \alpha_n U_n)$$

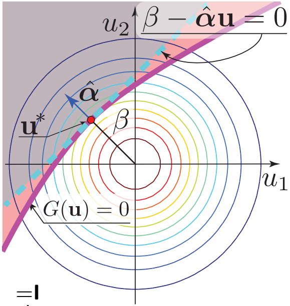
$$p_f = \Pr(\beta - \hat{\boldsymbol{\alpha}}\mathbf{u} \leq 0)$$



$$\frac{Z}{\|\nabla G(\mathbf{u}^*)\|} \cong \beta - \hat{\boldsymbol{\alpha}}\mathbf{U} = \beta - (\alpha_1 U_1 + \ldots + \alpha_n U_n)$$

et comme cette marge de sécurité est linéaire

$$\mathbb{E}\left[\frac{Z}{\|\nabla G(\mathbf{u}^*)\|}\right] \cong \beta, \quad (\mathbf{M_Y} = \overbrace{\mathbf{B}}^{=\beta} + \overbrace{\mathbf{A}}^{=\hat{\boldsymbol{\alpha}}} \overbrace{\mathbf{M_X}}^{\mathbb{E}[\mathbf{U}]=\mathbf{0}})$$




$$\text{Var}\left[\frac{Z}{\|\nabla G(\mathbf{u}^*)\|}\right] \cong \alpha_1^2 + \ldots + \alpha_n^2 = 1, \quad (\mathbf{\Sigma_{YY}} = \mathbf{A}\overbrace{\mathbf{\Sigma_{XX}}}^{=\mathbf{I}}\mathbf{A}^\intercal)$$

$\alpha_i^2$ correspond à la contribution de $U_i$ à la variance de la marge de sécurité normalisée, i.e. **mesure d'importance**

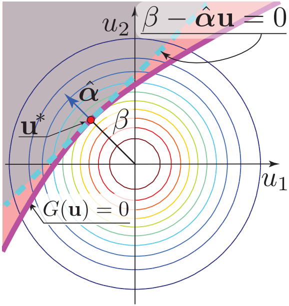
# Mesure d'importance: $\alpha_i$

$$\frac{Z}{\|\nabla G(\mathbf{u}^*)\|} \cong \beta - \hat{\boldsymbol{\alpha}}\mathbf{U} = \beta - (\alpha_1 U_1 + \ldots + \alpha_n U_n)$$

**Si $X_i$ et $X_j$ sont indépendants $\forall i \neq j$**

$\alpha_i^2$ correspond à l'importance relative de $X_i$ à la variance de la marge de sécurité normalisée.

Une valeur négative (positive) de $\alpha_i$ indique que $X_i$ contribue à la capacité (demande).



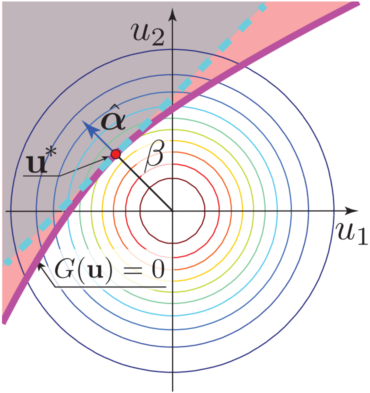
# Exemple – $\alpha_i$

$$\hat{\boldsymbol{\alpha}} = [\underbrace{\alpha_1}_{<0}, \underbrace{\alpha_2}_{>0}]$$

$\alpha_1 < 0$: Capacité

$\alpha_2 > 0$: Demande

$|\alpha_1| = |\alpha_2| \to \text{Imp}(X_1) = \text{Imp}(X_2)$



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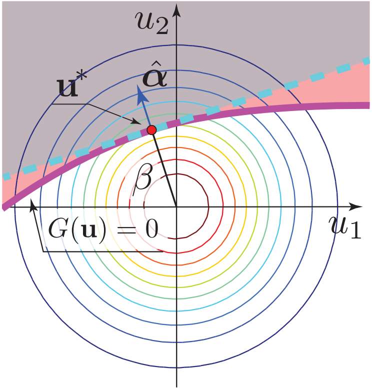
$$\hat{\boldsymbol{\alpha}} = [\underbrace{\alpha_1}_{<0}, \underbrace{\alpha_2}_{>0}]$$

$\alpha_1 < 0$: Capacité

$\alpha_2 > 0$: Demande

$|\alpha_1| < |\alpha_2| \rightarrow$ Imp($X_1$)<Imp($X_2$)



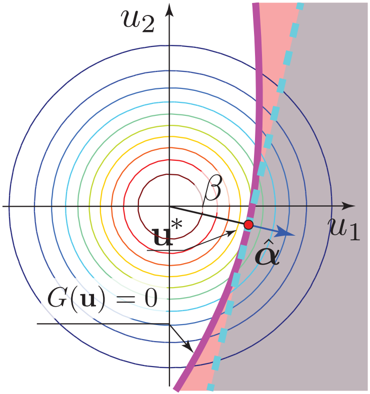
# Exemple – $\alpha_i$

$$\hat{\boldsymbol{\alpha}} = [\underbrace{\alpha_1}_{>0}, \underbrace{\alpha_2}_{<0}]$$

$\alpha_1 > 0$: Demande

$\alpha_2 < 0$: Capacité

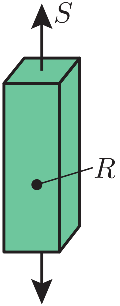
$|\alpha_1| > |\alpha_2| \to$ Imp($X_1$)>Imp($X_2$)



# Exemple #7.1 (Ref: Ex. #6.2)

Soit $X_1 = D \sim \ln\mathcal{N}(\mu_{\ln D}, \sigma_{\ln D})$ et $X_2 = S \sim \text{Gum}(\mu_S, \beta_S)$ où:

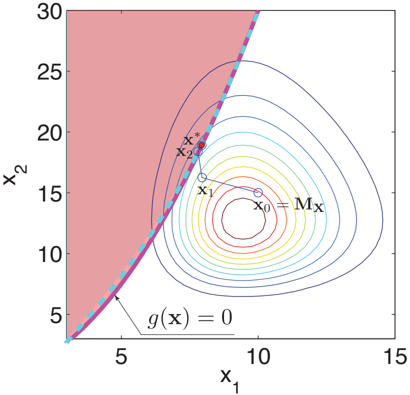
$$D\left\{\begin{array}{lcl}\mu_D & = & 10\,\text{mm}\\ \sigma_D & = & 2\,\text{mm}\end{array}\right., S\left\{\begin{array}{lcl}\mu_S & = & 15\,\text{kN}\\ \sigma_S & = & 5\,\text{kN}\end{array}\right., f_y = 300\,MPa,\ D \perp\!\!\!\perp S$$

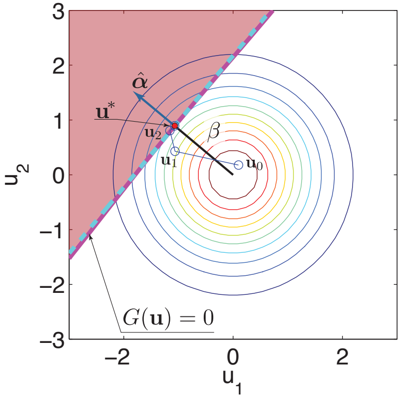



$$R(\mathbf{X}) = D^2 f_y,\ g(\mathbf{X}) = R(\mathbf{X}) - S$$

$$\mathbf{X} = \left\{\begin{array}{c} D \\ S \end{array}\right\},\ \mathbf{M_X} = \left\{\begin{array}{c} 10 \\ 15 \end{array}\right\}$$

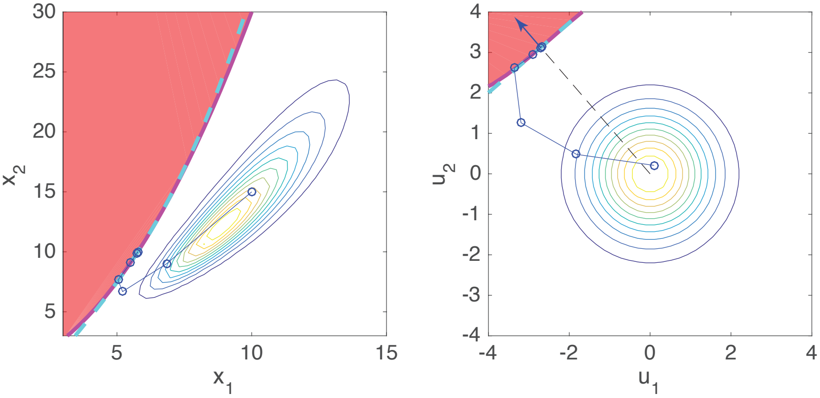
# Exemple #7.1 (Ref: Ex. #6.2) - Résultats [.m]




$$\frac{Z}{\|\nabla G(\mathbf{u}^*)\|} \cong \beta - \hat{\boldsymbol{\alpha}}\mathbf{u} = \beta - (\alpha_1 U_1 + \ldots + \alpha_n U_n), \quad \hat{\boldsymbol{\alpha}} = [-0.77, 0.64]$$

# Qu'arrive-t-il si $\rho \neq 0$? [.m]



# $X_i$ **dépendants** [.m]

Qu'arrive-t-il si les $X_i$ sont statistiquement dépendants?

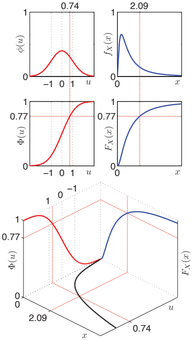
Il n'y a plus de correspondance directe entre $\alpha_i^2$ et l'importance de $X_i$ à la variance de la marge de sécurité normalisée.

**Solution**: linéariser la transformation $\mathbf{u} = \mathbf{T}(\mathbf{x})$ au point $\mathbf{u}^*$

$$\begin{aligned}\mathbf{u} = \mathbf{T}(\mathbf{x}) &\cong \mathbf{u}^* + \mathbf{J}_{\mathbf{u},\mathbf{x}}(\mathbf{x} - \mathbf{x}^*)\\ &= \mathbf{u}^* + \mathbf{J}_{\mathbf{u},\mathbf{x}}(\tilde{\mathbf{x}} - \mathbf{x}^*)\end{aligned}$$

où $\tilde{\mathbf{X}} \sim \mathcal{N}(\mathbf{M}_{\tilde{\mathbf{X}}}, \mathbf{\Sigma}_{\tilde{\mathbf{X}}\tilde{\mathbf{X}}})$ est une variable **aléatoire normale équivalente** telle que

$$\mathbf{M}_{\tilde{\mathbf{X}}} = \mathbf{x}^* - \mathbf{J}_{\mathbf{x},\mathbf{u}}\mathbf{u}^*, \quad \mathbf{\Sigma}_{\tilde{\mathbf{X}}\tilde{\mathbf{X}}} = \mathbf{J}_{\mathbf{u},\mathbf{x}}^{-1}\mathbf{I}(\mathbf{J}_{\mathbf{u},\mathbf{x}}^{-1})^\intercal = \mathbf{J}_{\mathbf{x},\mathbf{u}}\mathbf{J}_{\mathbf{x},\mathbf{u}}^\intercal$$



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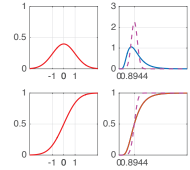
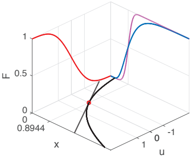
**Solution**: linéariser la transformation $\mathbf{u} = \mathbf{T}(\mathbf{x})$ au point $\mathbf{u}^*$

$$\begin{aligned}\mathbf{u} = \mathbf{T}(\mathbf{x}) &\cong \mathbf{u}^* + \mathbf{J}_{\mathbf{u},\mathbf{x}}(\mathbf{x} - \mathbf{x}^*)\\ &= \mathbf{u}^* + \mathbf{J}_{\mathbf{u},\mathbf{x}}(\tilde{\mathbf{x}} - \mathbf{x}^*)\end{aligned}$$




où $\tilde{\mathbf{X}} \sim \mathcal{N}(\mathbf{M}_{\tilde{\mathbf{X}}}, \mathbf{\Sigma}_{\tilde{\mathbf{X}}\tilde{\mathbf{X}}})$ est une variable **aléatoire normale équivalente** telle que




$$\mathbf{M}_{\tilde{\mathbf{X}}} = \mathbf{x}^* - \mathbf{J}_{\mathbf{x},\mathbf{u}}\mathbf{u}^*, \quad \mathbf{\Sigma}_{\tilde{\mathbf{X}}\tilde{\mathbf{X}}} = \mathbf{J}_{\mathbf{u},\mathbf{x}}^{-1}\mathbf{I}(\mathbf{J}_{\mathbf{u},\mathbf{x}}^{-1})^\intercal = \mathbf{J}_{\mathbf{x},\mathbf{u}}\mathbf{J}_{\mathbf{x},\mathbf{u}}^\intercal$$

$$
\begin{aligned}
\frac{Z}{\|\nabla G(\mathbf{u}^*)\|} &\cong \beta - \hat{\alpha}\mathbf{U} \\
&= \underbrace{\beta - \hat{\alpha}[\mathbf{u}^*}_{=0} + \mathbf{J}_{\mathbf{u},\mathbf{x}}(\tilde{\mathbf{X}} - \mathbf{x}^*)] \\
&= -\hat{\alpha}\mathbf{J}_{\mathbf{u},\mathbf{x}}(\tilde{\mathbf{X}} - \mathbf{x}^*)
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{E}\left[\frac{Z}{\|\nabla G(\mathbf{u}^*)\|}\right] &\cong -\hat{\alpha}\mathbf{J}_{\mathbf{u},\mathbf{x}}(\mathbf{M}_{\tilde{\mathbf{X}}} - \mathbf{x}^*) \\
&= -\hat{\alpha}\mathbf{J}_{\mathbf{u},\mathbf{x}}(\underbrace{\mathbf{x}^* - \mathbf{J}_{\mathbf{x},\mathbf{u}}\mathbf{u}^*}_{\mathbf{M}_{\tilde{\mathbf{X}}}} - \mathbf{x}^*) \\
&= \hat{\alpha}\mathbf{J}_{\mathbf{u},\mathbf{x}}(\mathbf{J}_{\mathbf{x},\mathbf{u}}\mathbf{u}^*) \\
&= \hat{\alpha}\mathbf{u}^* = \beta
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Var}\left[\frac{Z}{\|\nabla G(\mathbf{u}^*)\|}\right] &\cong \hat{\alpha}\mathbf{J}_{\mathbf{u},\mathbf{x}}\boldsymbol{\Sigma}_{\tilde{\mathbf{X}}\tilde{\mathbf{X}}}\mathbf{J}_{\mathbf{u},\mathbf{x}}^{\intercal}\hat{\alpha}^{\intercal} \\
&= \underbrace{\hat{\alpha}\mathbf{J}_{\mathbf{u},\mathbf{x}}\mathbf{D}_{\tilde{\mathbf{X}}}\mathbf{D}_{\tilde{\mathbf{X}}}\mathbf{J}_{\mathbf{u},\mathbf{x}}^{\intercal}\hat{\alpha}^{\intercal}}_{\mathrm{Imp}(\mathrm{Var}[\tilde{\mathbf{X}}]\to Z)} + \underbrace{\hat{\alpha}\mathbf{J}_{\mathbf{u},\mathbf{x}}(\boldsymbol{\Sigma}_{\tilde{\mathbf{X}}\tilde{\mathbf{X}}} - \mathbf{D}_{\tilde{\mathbf{X}}}\mathbf{D}_{\tilde{\mathbf{X}}})\mathbf{J}_{\mathbf{u},\mathbf{x}}^{\intercal}\hat{\alpha}^{\intercal}}_{\mathrm{Imp}(\mathrm{cov}[\tilde{\mathbf{X}}]\to Z)}
\end{aligned}
$$

# Mesure d'importance: $\hat{\gamma}$

$$\boxed{\hat{\gamma} = \frac{\hat{\alpha}\mathbf{J_{u,x}}\mathbf{D_{\tilde{X}}}}{\|\hat{\alpha}\mathbf{J_{u,x}}\mathbf{D_{\tilde{X}}}\|}}$$

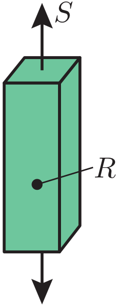
$\hat{\gamma}_i$ correspond à l'importance relative de $X_i$ par rapport à la variance de la marge de sécurité normalisé.

Une valeur négative (positive) de $\hat{\gamma}_i$ indique que $X_i$ contribue à la capacité (demande).

# Exemple #7.2 (Ref. Ex. 6.4)

Soit $X_1 = D \sim \ln\mathcal{N}(\mu_{\ln D}, \sigma_{\ln D})$ et $X_2 = S \sim \text{Gum}(\mu_S, \beta_S)$ où:

$$D\begin{cases}\mu_D &= 10\,\text{mm}\\ \sigma_D &= 2\,\text{mm}\end{cases} \quad S\begin{cases}\mu_S &= 15\,\text{kN}\\ \sigma_S &= 5\,\text{kN}\end{cases} \quad f_y = 300\ MPa, \ \rho_{D,S} = 0.30$$




$f_{\mathbf{X}}(\mathbf{x})$: distribution conjointe de type **Nataf**

$$R(\mathbf{X}) = D^2 f_y, \ \ g(\mathbf{X}) = R(\mathbf{X}) - S$$

$$\mathbf{X} = \left\{\begin{array}{c} D \\ S \end{array}\right\}, \ \mathbf{M_X} = \left\{\begin{array}{c} 10 \\ 15 \end{array}\right\}$$

$$\boxed{\mathbf{\Sigma}_{\tilde{\mathbf{X}}\tilde{\mathbf{X}}} = \mathbf{J}_{\mathbf{x},\mathbf{u}}\mathbf{J}_{\mathbf{x},\mathbf{u}}^{\intercal}, \quad [\mathbf{D}_{\tilde{\mathbf{X}}}]_{ii} = \sqrt{[\mathbf{\Sigma}_{\tilde{\mathbf{X}}\tilde{\mathbf{X}}}]_{ii}}}$$

# Densité de probabilité conjointe [ ]

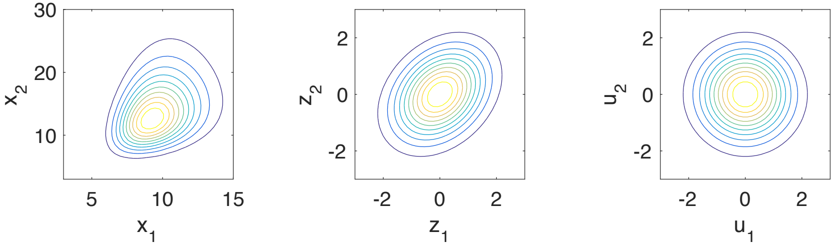
$$f_{\mathbf{X}}(\mathbf{x}) = \phi_n(\mathbf{z}, \mathbf{R}_0) \prod_{i=1}^{n} \frac{f_i(X_i)}{\phi(z_i)}$$

$$\begin{aligned}
\rho_{0,X_1X_2} &= \rho_{X_1X_2}(1.029 + 0.001\rho_{X_1X_2} + 0.014\delta_{X_1} + \ldots \\
&\quad +0.004\rho^2_{X_1X_2} + 0.233\delta^2_{X_1} - 0.197\rho_{X_1X_2}\delta_{X_1}) \\
&= 0.31
\end{aligned}$$

$$\mathbf{R}_0 = \begin{bmatrix} 1 & 0.31 \\ 0.31 & 1 \end{bmatrix}, \; \mathbf{L}_0 = \begin{bmatrix} 1 & 0 \\ 0.31 & 0.95 \end{bmatrix}$$

$$\mathbf{J}_{\mathbf{u},\mathbf{x}} = \mathbf{J}_{\mathbf{u},\mathbf{z}}\mathbf{J}_{\mathbf{z},\mathbf{x}} = \mathbf{L}_0^{-1}\,\text{diag}\left[\frac{f_{X_i}(x_i)}{\phi(z_i)}\right]$$

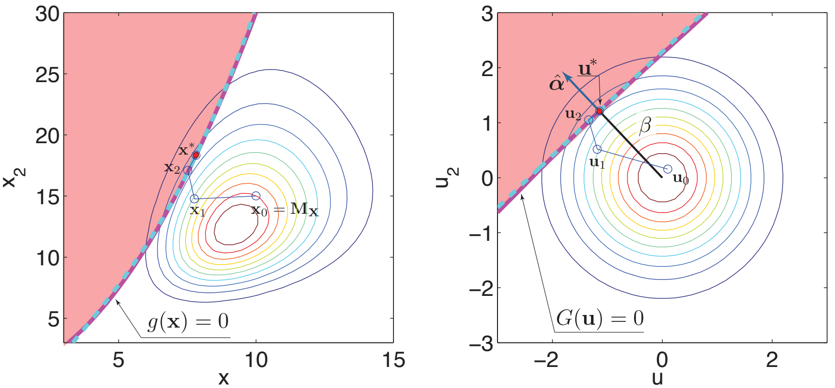
# Transformation $\mathbf{X} \leftrightarrow \mathbf{Z} \leftrightarrow \mathbf{U}$




$$\mathbf{x} = \begin{Bmatrix} x_1 \\ \vdots \\ x_n \end{Bmatrix} \rightarrow \mathbf{z} = \begin{Bmatrix} \Phi^{-1}[F_1(x_1)] \\ \dots \\ \Phi^{-1}[F_n(x_n)] \end{Bmatrix} \rightarrow \mathbf{u} = \mathbf{L}_0^{-1}\mathbf{z}$$

$$\mathbf{x} = \begin{Bmatrix} F_1^{-1}[\Phi(z_1)] \\ \dots \\ F_n^{-1}[\Phi(z_n)] \end{Bmatrix} \leftarrow \mathbf{z} = \mathbf{L}_0\mathbf{u} \leftarrow \mathbf{u} = \begin{Bmatrix} u_1 \\ \vdots \\ u_n \end{Bmatrix}$$

# Résultats [.m]



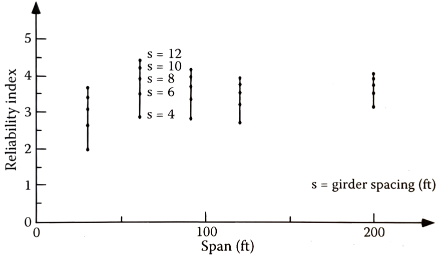

$$\hat{\gamma} = \frac{\hat{\alpha}\mathbf{J}_{\mathbf{u},\mathbf{x}}\mathbf{D}_{\tilde{\mathbf{X}}}}{\|\hat{\alpha}\mathbf{J}_{\mathbf{u},\mathbf{x}}\mathbf{D}_{\tilde{\mathbf{X}}}\|} = [-0.77, 0.64], \quad \hat{\alpha} = [-0.69, 0.73]$$

# Plan de la section

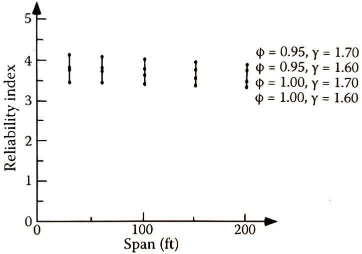
# Calibration des codes

$$\overbrace{\alpha_d d + \gamma\psi(\alpha_l l + \alpha_w w + \alpha_t t)}^{\text{Coefficients de pondérations – } \alpha., \gamma, \psi, \phi} < \phi_y f_y a$$

**Facteurs de sécurité / Contraintes admissibles** ($< 1990'$)




**LRFD / Coefficients de pondérations** ($> 1990'$)




Calibration de $\alpha., \gamma, \psi, \phi \rightarrow$ **même indice de fiabilité,** $\beta$

[Nowak and Collins (2012), *Reliability of structures*, CRC press.]

# Méthode de calcul des $\alpha., \gamma, \psi, \phi$

Si on fait l'hypothèse que les valeurs nominales $\bar{\mathbf{x}} = \mathbf{M_X}$

$$\phi_i = \frac{x_i^*}{[\mathbf{M_X}]_i} \rightarrow \boldsymbol{\phi}\bar{\mathbf{x}} = \underbrace{\mathbf{x}^* \leftrightarrow \mathbf{u}^*}_{||\mathbf{u}^*||=\beta}$$

Si $\beta \neq \beta_{\text{code}}$ ⚠ $\rightarrow \mathbf{M_X} + \boldsymbol{\Delta}$

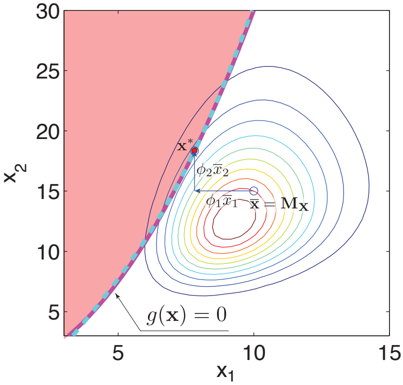
$$\boldsymbol{\Delta}_\mathbf{X} = \arg\min_{\boldsymbol{\Delta}} ||\boldsymbol{\Delta} : \{\beta = \beta_{\text{code}}\}||$$

Algorithme **iHL-RF modifié** [ ]

[1-6] Mêmes étapes

[7] $\mathbf{M_X} \leftarrow \mathbf{M_X} - (\beta_{\text{code}} - \beta_i)\lambda\boldsymbol{\alpha}$

[8] Recommencer à l'étape 2 avec $\mathbf{x}_{i+1} = \mathbf{M_X} + \mathbf{D_X}\mathbf{L}\mathbf{u}_{i+1}$



# Méthode de calcul des $\alpha., \gamma, \psi, \phi$

Si on fait l'hypothèse que les valeurs nominales $\bar{\mathbf{x}} = \mathbf{M_X}$

$$\phi_i = \frac{x_i^*}{[\mathbf{M_X}]_i} \rightarrow \boldsymbol{\phi}\bar{\mathbf{x}} = \underbrace{\mathbf{x}^* \leftrightarrow \mathbf{u}^*}_{||\mathbf{u}^*||=\beta}$$

Si $\beta \neq \beta_{\text{code}}$ ⚠ $\rightarrow \mathbf{M_X} + \boldsymbol{\Delta}$

$$\boldsymbol{\Delta}_\mathbf{X} = \arg\min_{\boldsymbol{\Delta}} ||\boldsymbol{\Delta} : \{\beta = \beta_{\text{code}}\}||$$

Algorithme **iHL-RF modifié** [ ]

[1-6] Mêmes étapes

[7] $\mathbf{M_X} \leftarrow \mathbf{M_X} - (\beta_{\text{code}} - \beta_i)\lambda\boldsymbol{\alpha}$

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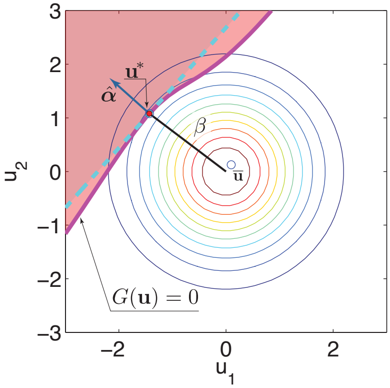


# Méthode de calcul des $\alpha., \gamma, \psi, \phi$

Si on fait l'hypothèse que les valeurs nominales $\bar{\mathbf{x}} = \mathbf{M_X}$

$$\phi_i = \frac{x_i^*}{[\mathbf{M_X}]_i} \to \phi\bar{\mathbf{x}} = \underbrace{\mathbf{x}^* \leftrightarrow \mathbf{u}^*}_{||\mathbf{u}^*||=\beta}$$

Si $\beta \neq \beta_{\text{code}}$ ⚠ $\to \mathbf{M_X} + \boldsymbol{\Delta}$

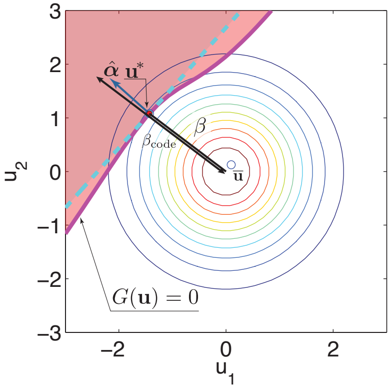
$$\boldsymbol{\Delta}_\mathbf{X} = \underset{\boldsymbol{\Delta}}{\arg\min} ||\boldsymbol{\Delta} : \{\beta = \beta_{\text{code}}\}||$$

Algorithme **iHL-RF modifié** [ ]

[1-6] Mêmes étapes

[7] $\mathbf{M_X} \leftarrow \mathbf{M_X} - (\beta_{\text{code}} - \beta_i)\lambda\boldsymbol{\alpha}$

[8] Recommencer à l'étape 2 avec $\mathbf{x}_{i+1} = \mathbf{M_X} + \mathbf{D_X}\mathbf{L}\mathbf{u}_{i+1}$



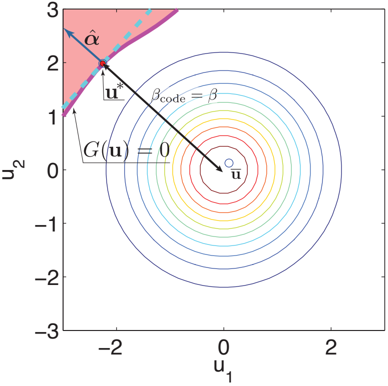
# Méthode de calcul des $\alpha.,\gamma,\psi,\phi$

Si on fait l'hypothèse que les valeurs nominales $\bar{\mathbf{x}} = \mathbf{M_X}$

$$\phi_i = \frac{x_i^*}{[\mathbf{M_X}]_i} \rightarrow \phi\bar{\mathbf{x}} = \underbrace{\mathbf{x}^* \leftrightarrow \mathbf{u}^*}_{||\mathbf{u}^*||=\beta}$$

Si $\beta \neq \beta_{\text{code}}$ ⚠ $\rightarrow \mathbf{M_X} + \boldsymbol{\Delta}$

$$\boldsymbol{\Delta}_\mathbf{X} = \arg\min_{\boldsymbol{\Delta}} ||\boldsymbol{\Delta} : \{\beta = \beta_{\text{code}}\}||$$




Algorithme **iHL-RF modifié** []

[1-6] Mêmes étapes

[7] $\mathbf{M_X} \leftarrow \mathbf{M_X} - (\beta_{\text{code}} - \beta_i)\lambda\boldsymbol{\alpha}$

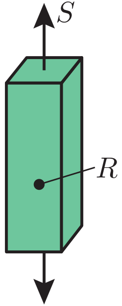
[8] Recommencer à l'étape 2 avec $\mathbf{x}_{i+1} = \mathbf{M_X} + \mathbf{D_X}\mathbf{L}\mathbf{u}_{i+1}$

# Exemple #7.3a (Ref: Ex. #6.4 []) []

Soit $X_1 = D \sim \ln\mathcal{N}(\mu_{\ln D}, \sigma_{\ln D})$ et $X_2 = S \sim \text{Gum}(\mu_S, \beta_S)$ où:

$f_{\mathbf{X}}(\mathbf{x})$: distribution conjointe de type **Nataf**

$$D\begin{cases}\mu_D &= 10\,\text{mm}\\ \sigma_D &= 2\,\text{mm}\end{cases} \quad S\begin{cases}\mu_S &= 15\,\text{kN}\\ \sigma_S &= 5\,\text{kN}\end{cases} \quad f_y = 300\ MPa,\ \rho_{D,S} = 0.30$$

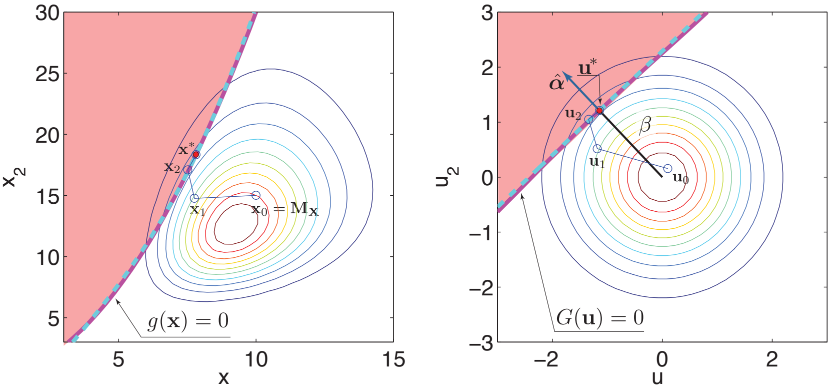



$$R(\mathbf{X}) = D^2 f_y,\ \ g(\mathbf{X}) = R(\mathbf{X}) - S$$

$$\mathbf{X} = \begin{Bmatrix} D \\ S \end{Bmatrix},\ \mathbf{M_X} = \begin{Bmatrix} 10 \\ 15 \end{Bmatrix}$$

**Quels devrait être $\phi_D$, $\phi_S$ si $\beta_{\text{cible}} = 1.67$?**

# Résultats [.m]



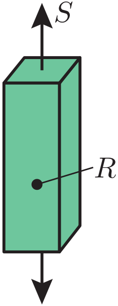

$$\beta = 1.67,\ \mathbf{x}^* = \left\{ \begin{array}{c} 7.82 \\ 18.34 \end{array} \right\},\ \mathbf{M_X} = \left\{ \begin{array}{c} 10 \\ 15 \end{array} \right\},\ \phi = \frac{\mathbf{x}^*}{\mathbf{M_X}} = \left\{ \begin{array}{c} 0.78 \\ 1.22 \end{array} \right\}$$

# Exemple #7.3b (Ref: Ex. #6.4 [ ]) [ ]

Soit $X_1 = D \sim \ln\mathcal{N}(\mu_{\ln D}, \sigma_{\ln D})$ et $X_2 = S \sim \text{Gum}(\mu_S, \beta_S)$ où:

$f_{\mathbf{X}}(\mathbf{x})$: distribution conjointe de type **Nataf**

$$D\left\{\begin{array}{lcl}\mu_D & = & 10\,\text{mm}\\ \sigma_D & = & 2\,\text{mm}\end{array}\right. \quad S\left\{\begin{array}{lcl}\mu_S & = & 15\,\text{kN}\\ \sigma_S & = & 5\,\text{kN}\end{array}\right. \quad f_y = 300\,MPa,\ \rho_{D,S} = 0.30$$

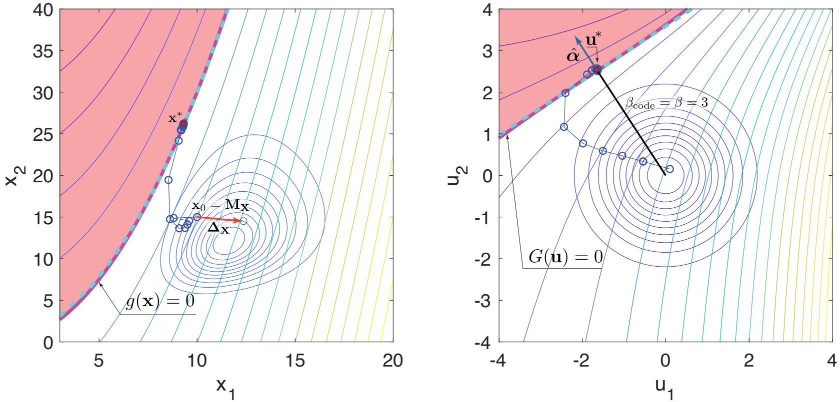



$$R(\mathbf{X}) = D^2 f_y,\ \ g(\mathbf{X}) = R(\mathbf{X}) - S$$

$$\mathbf{X} = \left\{\begin{array}{c} D \\ S \end{array}\right\},\ \mathbf{M_X} = \left\{\begin{array}{c} 10 \\ 15 \end{array}\right\}$$

Quels devrait être $\phi_D$, $\phi_S$ si $\beta_{\text{cible}} = \mathbf{3}$?

# Résultats [.m]



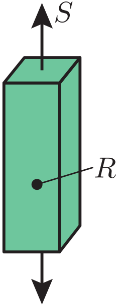

$$\beta = 3,\ \mathbf{x}^* = \left\{ \begin{array}{c} 9.3 \\ 25.8 \end{array} \right\},\ \mathbf{M_X} + \boldsymbol{\Delta}_\mathbf{X} = \left\{ \begin{array}{c} 12.3 \\ 14.5 \end{array} \right\},\ \phi = \left\{ \begin{array}{c} 0.76 \\ 1.79 \end{array} \right\}$$

# Exemple #7.3c (Ref: Ex. #6.4 [ ]) [ ]

Soit $X_1 = D \sim \ln\mathcal{N}(\mu_{\ln D}, \sigma_{\ln D})$ et $X_2 = S \sim \text{Gum}(\mu_S, \beta_S)$ où:

$f_{\mathbf{X}}(\mathbf{x})$: distribution conjointe de type **Nataf**

$$D\begin{cases}\mu_D &= 10\,\text{mm}\\ \sigma_D &= 2\,\text{mm}\end{cases}\quad S\begin{cases}\mu_S &= 15\,\text{kN}\\ \sigma_S &= 1.5\,\text{kN}\end{cases}\quad f_y = 300\,MPa,\ \rho_{D,S} = 0.0$$

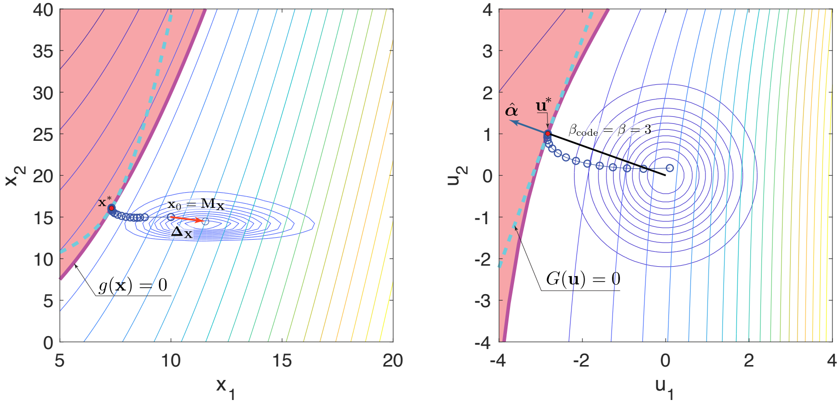



$$R(\mathbf{X}) = D^2 f_y,\ \ g(\mathbf{X}) = R(\mathbf{X}) - S$$

$$\mathbf{X} = \left\{\begin{array}{c} D \\ S \end{array}\right\},\ \mathbf{M_X} = \left\{\begin{array}{c} 10 \\ 15 \end{array}\right\}$$

Quels devraient être $\phi_D$, $\phi_S$ si $\beta_{\text{cible}} = \mathbf{3}$?

# Résultats [.m]




$$\beta = 3,\ \mathbf{x}^* = \left\{ \begin{array}{c} 7.3 \\ 16.1 \end{array} \right\},\ \mathbf{M_X} + \mathbf{\Delta_X} = \left\{ \begin{array}{c} 11.9 \\ 14.7 \end{array} \right\},\ \phi = \left\{ \begin{array}{c} 0.62 \\ 1.09 \end{array} \right\}$$

# Plan de la section

# Sensibilité de $p_f$ & $\beta$ par rapport à $\boldsymbol{\theta}$ []

Soit un vecteur de variables aléatoires $\mathbf{X} = \{X_1, \ldots, X_n\}^\intercal$ où $f_{\mathbf{X}}(\mathbf{x}; \boldsymbol{\theta}_f)$ est la densité de probabilité conjointe paramétrée par $\boldsymbol{\theta}_f$.

On cherche:

$$\left[\frac{\partial \beta}{\partial \theta_{f,1}}, \cdots, \frac{\partial \beta}{\partial \theta_{f,m}}\right] = \nabla_{\boldsymbol{\theta}_f} \beta$$

et

$$\left[\frac{\partial p_f}{\partial \theta_{f,1}}, \cdots, \frac{\partial p_f}{\partial \theta_{f,m}}\right] = \nabla_{\boldsymbol{\theta}_f} p_f$$

# Estimation de $\nabla_{\boldsymbol{\theta}_f}\beta$

## À partir d'une solution de type FORM

$$\boxed{\mathbf{u}^*(\theta) = \beta(\theta)\hat{\boldsymbol{\alpha}}(\theta), \text{ où } \hat{\boldsymbol{\alpha}}(\theta)\hat{\boldsymbol{\alpha}}(\theta)^\intercal = 1}$$

$$\begin{aligned}\frac{d[\hat{\boldsymbol{\alpha}}(\theta)\hat{\boldsymbol{\alpha}}(\theta)^\intercal]}{d\theta} &= \frac{d\hat{\boldsymbol{\alpha}}(\theta)}{d\theta}\hat{\boldsymbol{\alpha}}(\theta)^\intercal + \hat{\boldsymbol{\alpha}}(\theta)\frac{d\hat{\boldsymbol{\alpha}}(\theta)^\intercal}{d\theta}\\ &= 2\hat{\boldsymbol{\alpha}}(\theta)\frac{d\hat{\boldsymbol{\alpha}}(\theta)^\intercal}{d\theta} = 0\end{aligned}$$

$$\frac{d\mathbf{u}^*(\theta)}{d\theta} = \frac{d\beta(\theta)}{d\theta}\hat{\boldsymbol{\alpha}}(\theta)^\intercal + \beta(\theta)\frac{d\hat{\boldsymbol{\alpha}}(\theta)^\intercal}{d\theta}$$

$$\hat{\boldsymbol{\alpha}}(\theta)\frac{d\mathbf{u}^*(\theta)}{d\theta} = \frac{d\beta(\theta)}{d\theta}\overbrace{\hat{\boldsymbol{\alpha}}(\theta)\hat{\boldsymbol{\alpha}}(\theta)^\intercal}^{=1} + \beta(\theta)\overbrace{\frac{d\hat{\boldsymbol{\alpha}}(\theta)^\intercal}{d\theta}\hat{\boldsymbol{\alpha}}(\theta)}^{=0}$$

# Estimation de $\nabla_{\boldsymbol{\theta}_f}\beta$

À partir d'une solution de type FORM

$$\boxed{\mathbf{u}^*(\theta) = \beta(\theta)\hat{\boldsymbol{\alpha}}(\theta),\ \text{où } \hat{\boldsymbol{\alpha}}(\theta)\hat{\boldsymbol{\alpha}}(\theta)^\intercal = 1}$$

$$\begin{aligned}\frac{d[\hat{\boldsymbol{\alpha}}(\theta)\hat{\boldsymbol{\alpha}}(\theta)^\intercal]}{d\theta} &= \frac{d\hat{\boldsymbol{\alpha}}(\theta)}{d\theta}\hat{\boldsymbol{\alpha}}(\theta)^\intercal + \hat{\boldsymbol{\alpha}}(\theta)\frac{d\hat{\boldsymbol{\alpha}}(\theta)^\intercal}{d\theta}\\ &= 2\hat{\boldsymbol{\alpha}}(\theta)\frac{d\hat{\boldsymbol{\alpha}}(\theta)^\intercal}{d\theta} = 0\end{aligned}$$

$$\frac{d\mathbf{u}^*(\theta)}{d\theta} = \frac{d\beta(\theta)}{d\theta}\hat{\boldsymbol{\alpha}}(\theta)^\intercal + \beta(\theta)\frac{d\hat{\boldsymbol{\alpha}}(\theta)^\intercal}{d\theta}$$

$$\boxed{\frac{d\beta(\theta)}{d\theta} = \hat{\boldsymbol{\alpha}}(\theta)\frac{d\mathbf{u}^*(\theta)}{d\theta}}$$

# Estimation de $\nabla_{\boldsymbol{\theta}_f}\beta$ (cont.)

Soit la transformation

$$\mathbf{u}(\boldsymbol{\theta}) = \mathbf{T}(\mathbf{x}, \boldsymbol{\theta}),\ \text{où}\ \mathbf{u}^*(\boldsymbol{\theta}) = \mathbf{T}(\mathbf{x}^*, \boldsymbol{\theta})$$

$$\boxed{\left[\frac{\partial\beta}{\partial\theta_{f,1}}, \cdots, \frac{\partial\beta}{\partial\theta_{f,m}}\right] = \nabla_{\boldsymbol{\theta}_f}\beta = \hat{\boldsymbol{\alpha}}\mathbf{J}_{\mathbf{u},\boldsymbol{\theta}_f}(\mathbf{x}^*, \boldsymbol{\theta}_f)}$$

où,

$$\mathbf{J}_{\mathbf{u},\boldsymbol{\theta}_f} = \mathbf{J}_{\mathbf{u},\boldsymbol{\theta}_f}(\mathbf{x}^*, \boldsymbol{\theta}_f) = \left[\begin{array}{ccc} \frac{d\,\overbrace{T_{u_1}(\mathbf{x}^*, \boldsymbol{\theta})}^{\mathbf{u}_1(\boldsymbol{\theta})}}{d\theta_1} & \cdots & \frac{dT_{u_1}(\mathbf{x}^*,\boldsymbol{\theta})}{d\theta_m} \\ \vdots & \ddots & \vdots \\ \frac{dT_{u_n}(\mathbf{x}^*,\boldsymbol{\theta})}{d\theta_1} & \cdots & \frac{dT_{u_n}(\mathbf{x}^*,\boldsymbol{\theta})}{d\theta_m} \end{array}\right]$$

$$\boxed{\left[\frac{\partial p_f}{\partial\theta_{f,1}}, \cdots, \frac{\partial p_f}{\partial\theta_{f,m}}\right] = \nabla_{\boldsymbol{\theta}_f} p_f = -\phi(\beta)\nabla_{\boldsymbol{\theta}_f}\beta}$$

# Jacobien $\mathbf{J}_{\mathbf{u},\boldsymbol{\theta}_f}(\mathbf{x}^*, \boldsymbol{\theta}_f)$

Pour la famille de distribution **Nataf,**

$$\boxed{\mathbf{J}_{\mathbf{u},\boldsymbol{\theta}_f} = \mathbf{L}_0^{-1}\begin{bmatrix} \frac{1}{\phi(z_1^*)}\frac{dF_{X_1}(x_1^*,\theta_1)}{d\theta_1} & \cdots & \frac{1}{\phi(z_1^*)}\frac{dF_{X_1}(x_1^*,\theta_m)}{d\theta_m} \\ \vdots & \ddots & \vdots \\ \frac{1}{\phi(z_n^*)}\frac{dF_{X_n}(x_n^*,\theta_1)}{d\theta_1} & \cdots & \frac{1}{\phi(z_n^*)}\frac{dF_{X_n}(x_n^*,\theta_m)}{d\theta_m} \end{bmatrix}}$$

**Note**: Les dérivés $\frac{dF_{X_i}(x_i^*,\theta_j)}{d\theta_j}$ peuvent être calculés numériquement. $(\Delta\theta_j \approx \theta_j \times 10^{-6})$

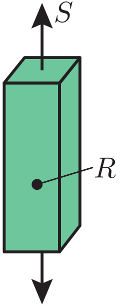
$$\frac{dF_{X_i}(x_i^*,\theta_j)}{d\theta_j} \cong \frac{F_{X_i}(x_i^*,\theta_j + \Delta\theta_j) - F_{X_i}(x_i^*,\theta_j)}{\Delta\theta_j}$$

# Exemple #7.3 (Ref. Ex. 6.4)

Soit $X_1 = D \sim \ln\mathcal{N}(\mu_{\ln D}, \sigma_{\ln D})$ et $X_2 = S \sim \text{Gum}(\mu_S, \beta_S)$ où:

$f_{\mathbf{X}}(\mathbf{x})$: distribution conjointe de type **Nataf**

$$D\begin{cases}\mu_D &= 10\,\text{mm}\\ \sigma_D &= 2\,\text{mm}\end{cases} \quad S\begin{cases}\mu_S &= 15\,\text{kN}\\ \sigma_S &= 5\,\text{kN}\end{cases} \quad f_y = 300\,MPa,\ \rho_{D,S} = 0.30$$




$$R(\mathbf{X}) = D^2 f_y,\ g(\mathbf{X}) = R(\mathbf{X}) - S$$

$$\boldsymbol{\theta}_f = [\mu_{\ln D}, \sigma_{\ln D}, \mu_S, \beta_S]^\intercal$$

$$\boxed{\nabla_{\boldsymbol{\theta}_f}\beta?}$$

# Transformation $\mathbf{X} \leftrightarrow \mathbf{Z} \leftrightarrow \mathbf{U}$ et Jacobien

$$\mathbf{u} = \mathbf{L}_0^{-1}\mathbf{z}, \text{ où } \mathbf{z} = \left\{ \begin{array}{c} \Phi^{-1}[F_1(x_1)] \\ \dots \\ \Phi^{-1}[F_n(x_n)] \end{array} \right\}$$

$$\mathbf{x} = \left\{ \begin{array}{c} F_1^{-1}[\Phi(z_1)] \\ \dots \\ F_n^{-1}[\Phi(z_n)] \end{array} \right\}, \text{ où } \mathbf{z} = \mathbf{L}_0\mathbf{u}$$

$$\mathbf{J}_{\mathbf{u},\mathbf{x}} = \mathbf{J}_{\mathbf{u},\mathbf{z}}\mathbf{J}_{\mathbf{z},\mathbf{x}} = \mathbf{L}_0^{-1} \operatorname{diag}\left[\frac{f_{X_i}(x_i)}{\phi(z_i)}\right]$$

$$\mathbf{J}_{\mathbf{u},\boldsymbol{\theta}_f} = \mathbf{L}_0^{-1} \left[ \begin{array}{cccc} \frac{1}{\phi(z_1^*)}\frac{dF_{X_1}(x_1^*,\theta_1)}{d\theta_1} & \frac{1}{\phi(z_1^*)}\frac{dF_{X_1}(x_1^*,\theta_2)}{d\theta_2} & 0 & 0 \\ 0 & 0 & \frac{1}{\phi(z_2^*)}\frac{dF_{X_2}(x_2^*,\theta_3)}{d\theta_3} & \frac{1}{\phi(z_2^*)}\frac{dF_{X_2}(x_2^*,\theta_4)}{d\theta_4} \end{array} \right]$$

# Résultats [.m]

$$\nabla_{\boldsymbol{\theta}_f}\beta = \hat{\alpha}\mathbf{J}_{\mathbf{u},\boldsymbol{\theta}_f}(\mathbf{x}^*, \boldsymbol{\theta}_f) = [4.7, -5.4, -0.13, -0.18]$$

$$\nabla_{\boldsymbol{\theta}_f} p_f = -\phi(\beta)\nabla_{\boldsymbol{\theta}_f}\beta = [-0.46, 0.53, 0.01, 0.02]$$

⚠ Les sensibilités par rapport aux paramètres des distributions sont souvent peu intuitives.
**Solution**: Sensibilité par rapport à $\mathbf{M_X}$ et $\mathbf{D_X}$ ⚠

# Plan de la section

# Sensibilité par rapport à $\mathbf{M_X}$ et $\mathbf{D_X}$, $\nabla_{[\mathbf{M_X},\mathbf{D_X}]}\beta$

À l'exception de la loi normale, la plupart des densités de probabilité ne sont par explicitement paramétrisé par $\mathbf{M_X}$ et $\mathbf{D_X}$.

Comment estime-t-on $\nabla_{[\mathbf{M_X},\mathbf{D_X}]}\beta$?

$$\mu_{X_i} = \text{fct}(\boldsymbol{\theta}_f)$$
$$\sigma_{X_i} = \text{fct}(\boldsymbol{\theta}_f)$$

$$\boxed{\nabla_{[\mathbf{M_X},\mathbf{D_X}]}\beta = \nabla_{\boldsymbol{\theta}_f}\beta \mathbf{J}_{\boldsymbol{\theta}_f,[\mathbf{M_X},\mathbf{D_X}]}}$$

# Normalisation de $\nabla_{[\mathbf{M_X},\mathbf{D_X}]}\beta$

Sensibilité par rapport aux valeurs moyennes

$$\left[\frac{\partial\beta}{\partial\mu_{X_1}}, \cdots, \frac{\partial\beta}{\partial\mu_{X_n}}\right] = \nabla_{\mathbf{M_X}}\beta$$

Sensibilité par rapport aux valeurs d'écarts types

$$\left[\frac{\partial\beta}{\partial\sigma_{X_1}}, \cdots, \frac{\partial\beta}{\partial\sigma_{X_n}}\right] = \nabla_{\mathbf{D_X}}\beta$$

Mesures de sensibilité **adimensionnelles**

$$\boxed{\boldsymbol{\delta} = \nabla_{\mathbf{M_X}}\beta\mathbf{D_X}} = \left[\frac{\partial\beta}{\partial\mu_{X_1}}\sigma_{X_1}, \cdots, \frac{\partial\beta}{\partial\mu_{X_n}}\sigma_{X_n}\right]$$

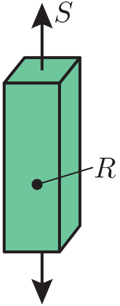
$$\boxed{\boldsymbol{\eta} = \nabla_{\mathbf{D_X}}\beta\mathbf{D_X}} = \left[\frac{\partial\beta}{\partial\sigma_{X_1}}\sigma_{X_1}, \cdots, \frac{\partial\beta}{\partial\sigma_{X_n}}\sigma_{X_n}\right]$$

# Exemple #7.3b (Ref. Ex. 6.4)

Soit $X_1 = D \sim \ln\mathcal{N}(\mu_{\ln D}, \sigma_{\ln D})$ et $X_2 = S \sim \text{Gum}(\mu_S, \beta_S)$ où:

$f_{\mathbf{X}}(\mathbf{x})$: distribution conjointe de type **Nataf**

$$D \begin{cases} \mu_D &= 10\,\text{mm} \\ \sigma_D &= 2\,\text{mm} \end{cases} \quad S \begin{cases} \mu_S &= 15\,\text{kN} \\ \sigma_S &= 5\,\text{kN} \end{cases} \quad f_y = 300\;MPa,\; \rho_{D,S} = 0.30$$




$$R(\mathbf{X}) = D^2 f_y,\;\; g(\mathbf{X}) = R(\mathbf{X}) - S$$

$$\boldsymbol{\theta}_f = [\mu_{\ln D}, \sigma_{\ln D}, \mu_S, \beta_S]^\intercal$$

$$\boxed{\nabla_{[\mathbf{M_X}, \boldsymbol{\Sigma}_{\mathbf{XX}}]}\beta?,\;\; \boldsymbol{\delta}?,\;\; \boldsymbol{\eta}?}$$

# Transformation $[\mathbf{M_X}, \mathbf{\Sigma_{XX}}] \to \boldsymbol{\theta}_f$

$X_1$: Densité de probabilité log normale

$$\mu_{\ln X_1} = \ln \mu_{X_1} - \frac{\sigma^2_{\ln X_1}}{2} = 2.28$$

$$\sigma_{\ln X_1} = \sqrt{\ln(1 + \delta^2_{X_1})} = 0.20$$

$X_2$: Densité de probabilité Gumbel (GEV, Type-I)

$$\left.\begin{array}{rcl} \mu & = & \mu_{X_2} - \beta\gamma \\ \beta & = & \dfrac{\sqrt{6}\sigma_{X_2}}{\pi} \end{array}\right\} \to \begin{array}{rcl} \mu & = & 12.8 \\ \beta & = & 3.9 \end{array}$$

# Transformation $[\mathbf{M_X}, \mathbf{D_X}] \to \boldsymbol{\theta}_f$ (cont.)

$$\boldsymbol{\theta}_f = \left\{ \begin{array}{c} \mu_{\ln X_1} \\ \sigma_{\ln X_1} \\ \mu \\ \beta \end{array} \right\} = \left\{ \begin{array}{c} \ln \mu_{X_1} - \dfrac{\sigma^2_{\ln X_1}}{2} \\ \sqrt{\ln(1+\delta^2_{X_1})} \\ \mu_{X_2} - 0.577\dfrac{\sqrt{6}\sigma_{X_2}}{\pi} \\ \dfrac{\sqrt{6}\sigma_{X_2}}{\pi} \end{array} \right\}, \quad \left\{ \begin{array}{c} \mathbf{M_X} \\ \mathbf{D_X} \end{array} \right\} = \left\{ \begin{array}{c} \mu_{X_1} \\ \mu_{X_2} \\ \sigma_{X_1} \\ \sigma_{X_2} \end{array} \right\}$$

$$\mathbf{J}_{\boldsymbol{\theta}_f,[\mathbf{M_X},\mathbf{D_X}]} = \begin{bmatrix} \frac{\partial \mu_{\ln X_1}}{\partial \mu_{X_1}} & \frac{\partial \mu_{\ln X_1}}{\partial \mu_{X_2}} & \frac{\partial \mu_{\ln X_1}}{\partial \sigma_{X_1}} & \frac{\partial \mu_{\ln X_1}}{\partial \sigma_{X_2}} \\ \frac{\partial \sigma_{\ln X_1}}{\partial \mu_{X_1}} & \frac{\partial \sigma_{\ln X_1}}{\partial \mu_{X_2}} & \frac{\partial \sigma_{\ln X_1}}{\partial \sigma_{X_1}} & \frac{\partial \sigma_{\ln X_1}}{\partial \sigma_{X_2}} \\ \frac{\partial \mu}{\partial \mu_{X_1}} & \frac{\partial \mu}{\partial \mu_{X_2}} & \frac{\partial \mu}{\partial \sigma_{X_1}} & \frac{\partial \mu}{\partial \sigma_{X_2}} \\ \frac{\partial \beta}{\partial \mu_{X_1}} & \frac{\partial \beta}{\partial \mu_{X_2}} & \frac{\partial \beta}{\partial \sigma_{X_1}} & \frac{\partial \beta}{\partial \sigma_{X_2}} \end{bmatrix}$$

# Transformation $[\mathbf{M_X}, \mathbf{D_X}] \to \boldsymbol{\theta}_f$ (cont.)

$$\boldsymbol{\theta}_f = \left\{\begin{array}{c} \mu_{\ln X_1} \\ \sigma_{\ln X_1} \\ \mu \\ \beta \end{array}\right\} = \left\{\begin{array}{c} \ln \mu_{X_1} - \dfrac{\sigma^2_{\ln X_1}}{2} \\ \sqrt{\ln(1+\delta^2_{X_1})} \\ \mu_{X_2} - 0.577\dfrac{\sqrt{6}\sigma_{X_2}}{\pi} \\ \dfrac{\sqrt{6}\sigma_{X_2}}{\pi} \end{array}\right\}, \quad \left\{\begin{array}{c} \mathbf{M_X} \\ \mathbf{D_X} \end{array}\right\} = \left\{\begin{array}{c} \mu_{X_1} \\ \mu_{X_2} \\ \sigma_{X_1} \\ \sigma_{X_2} \end{array}\right\}$$

$$\mathbf{J}_{\boldsymbol{\theta}_f, [\mathbf{M_X}, \mathbf{D_X}]} = \left[\begin{array}{cccc} \frac{1+2\delta^2_{X_1}}{\mu_{X_1}(1+\delta^2_{X_1})} & 0 & -\frac{\delta_{X_1}}{\mu_{X_1}(1+\delta^2_{X_1})} & 0 \\ -\frac{\delta^2_{X_1}}{\sigma_{\ln X_1}\mu_{X_1}(1+\delta^2_{X_1})} & 0 & \frac{\delta_{X_1}}{\sigma_{\ln X_1}\mu_{X_1}(1+\delta^2_{X_1})} & 0 \\ 0 & 1 & 0 & -\frac{0.577\sqrt{6}}{\pi} \\ 0 & 0 & 0 & \frac{\sqrt{6}}{\pi} \end{array}\right]$$

# Résultats [.m]

$$\{\mu_D, \mu_S, \sigma_D, \sigma_S\}$$

$$\begin{aligned}\nabla_{[\mathbf{M_X},\mathbf{D_X}]}\beta &= \nabla_{\boldsymbol{\theta}_f}\beta\mathbf{J}_{\boldsymbol{\theta}_f,[\mathbf{M_X},\mathbf{D_X}]}\\ &= [0.59, -0.13, -0.61, -0.09]\end{aligned}$$

$$\begin{aligned}\nabla_{[\mathbf{M_X},\mathbf{D_X}]}p_f &= -\phi(\beta)\nabla_{[\mathbf{M_X},\mathbf{D_X}]}\beta\\ &= [-0.06, 0.01, 0.006, 0.009]\end{aligned}$$

$$\boldsymbol{\delta} = \nabla_{\mathbf{M_X}}\beta\mathbf{D_X} = [1.2, -0.64]$$

$$\boldsymbol{\eta} = \nabla_{\mathbf{D_X}}\beta\mathbf{D_X} = [-1.2, -0.43]$$

# Résumé: Mesures d'importance et de sensibilité

- Imp($X_i \to Z$): Importance relative de $X_i$ à la variance de la marge de sécurité normalisée.

$$\boxed{\hat{\gamma} = \frac{\hat{\alpha}\mathbf{J}_{\mathbf{u},\mathbf{x}}\mathbf{D}_{\tilde{\mathbf{X}}}}{\|\hat{\alpha}\mathbf{J}_{\mathbf{u},\mathbf{x}}\mathbf{D}_{\tilde{\mathbf{X}}}\|}}$$

- Imp($\theta \to p_f$): Sensibilité de $\beta$ et $p_f$ par rapport aux paramètres $\theta_f$ de la distribution conjointe de **X**.

$$\boxed{\nabla_{\boldsymbol{\theta}_f}\beta = \hat{\alpha}\mathbf{J}_{\mathbf{u},\boldsymbol{\theta}_f}(\mathbf{x}^*, \boldsymbol{\theta}_f)}$$

- Imp($[\mathbf{M_X}, \mathbf{D_X}] \to p_f$): Sensibilité $\beta$ et $p_f$ par rapport aux valeurs moyennes et aux écarts types de la distribution conjointe de **X**.

$$\boxed{\nabla_{[\mathbf{M_X},\mathbf{D_X}]}\beta = \nabla_{\boldsymbol{\theta}_f}\beta\mathbf{J}_{\boldsymbol{\theta}_f,[\mathbf{M_X},\mathbf{D_X}]}}$$

Calibration des codes

$$\phi_i = \frac{x_i^*}{[\mathbf{M_X}]_i}$$

Mesures de sensibilité adimensionnelles

$$\boxed{\boldsymbol{\delta} = \nabla_{\mathbf{M_X}}\beta\mathbf{D_X}}$$

$$\boxed{\boldsymbol{\eta} = \nabla_{\mathbf{D_X}}\beta\mathbf{D_X}}$$

Avec ces mesures de sensibilité on peut estimer

$$\boxed{\beta_{\text{nouveau}} = \beta_{\text{ancien}} + \sum_i \frac{\partial\beta}{\partial\theta_i}\Delta\theta_i}$$

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