Module #5
Estimation de la fiabilité:
Approximation linéaire, sachant $\mathbf{M_X}$ et $\mathbf{\Sigma_{XX}}$

(CIV8530 - Fiabilité des structures et systèmes)

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Sections 5.1-5.2 - A. Der Kiureghian, (2014), *Structural and System Reliability (manuscript)*. Department of Civil and Environmental Engineering, UC Berkeley

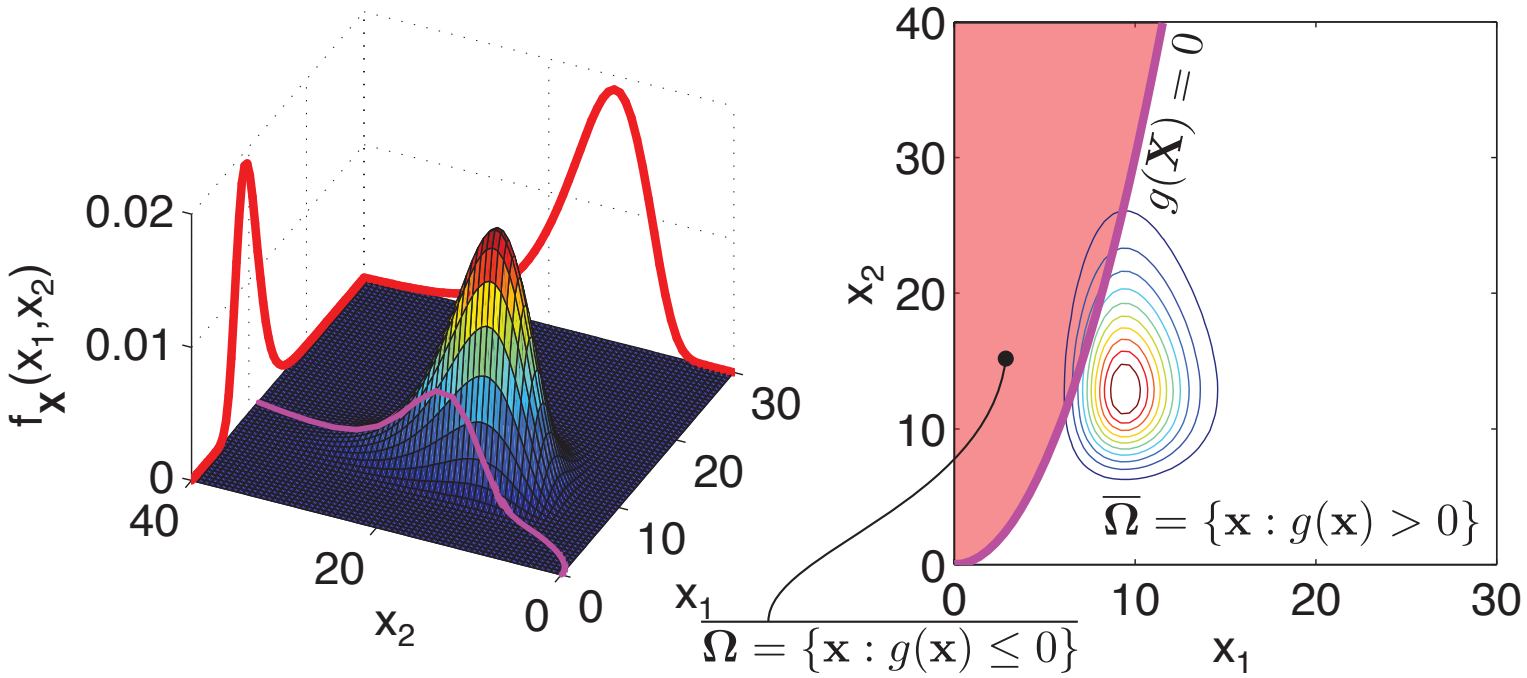
Probabilité de défaillance - Fonction d'état limite

$\mathbf{X} = \{X_1, X_2, \cdots, X_N\}$: ensemble de variables aléatoires

$f_{\mathbf{X}}(\mathbf{x})$: densité de probabilité conjointe

$g(\mathbf{X})$: fonction d'état limite

$g(\mathbf{X}) = 0$: surface d'état limite



Problématique?

Que fait-on lorsque $\mathbf{X} = \{X_1, X_2, \cdots, X_N\}$ et $N > 2$?

- Il n'est plus possible de tracer $f_{\mathbf{X}}(\mathbf{x})$
- Intégrer analytiquement en $N > 2$ dimensions est difficile

Solution: *Approximations analytiques, i.e.* **MCFOSM**, **FOSM**, *FORM, SORM* et *intégration numérique, i.e. Monte Carlo*

MCFOSM & FOSM

Moments: $\mu = \mathbb{E}[X]$, $\sigma^2 = \mathbb{E}[(X-\mu)^2]$

MCFOSM: Approximation du 1er ordre avec moments du 2eme ordre, centré sur $\mathbf{M_X}$
(Mean-Centered First-Order Second Moments)

FOSM: Approximation du 1er ordre avec moments du 2eme ordre, (First-Order Second Moments)

Avec **MCFOSM** *&* **FOSM**, *on assume qu'on ne connait que les deux premiers moments de* $f_{\mathbf{X}}(\mathbf{x})$, $\mathbf{X} \sim (\mathbf{M_X}, \mathbf{\Sigma_{XX}})$

$\mathbf{X} \sim (\mathbf{M_X}, \mathbf{\Sigma_{XX}})$

Si on ne connait que les deux premiers moments:

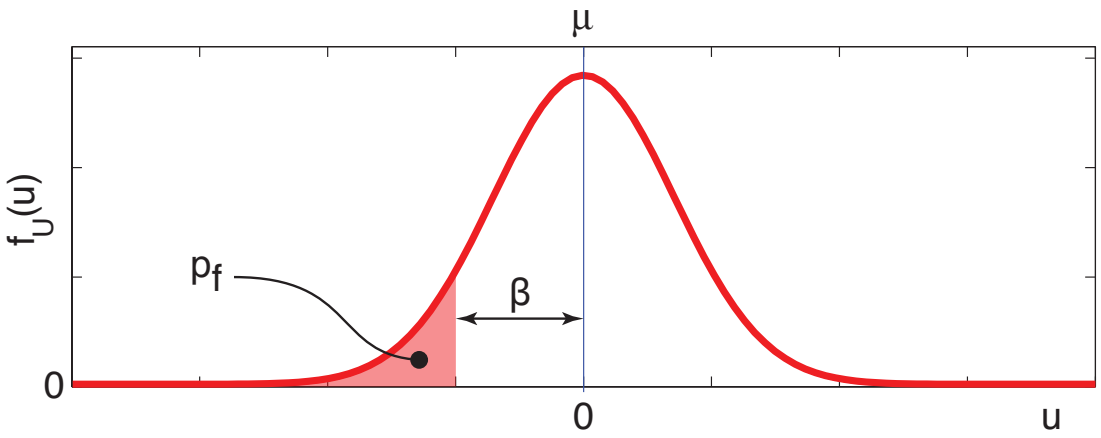
$$\mathbf{X} \sim ?(\underbrace{\mathbf{M_X}}_{1^{er}}, \underbrace{\mathbf{\Sigma_{XX}}}_{2^{eme}}) \approx \underbrace{\mathbf{X} \sim \mathcal{N}(\mathbf{M_X}, \mathbf{\Sigma_{XX}})}_{\text{Loi normale}}$$

Problématique? (cont.)

Soit $\mathbf{X} \sim (\mathbf{M_X}, \mathbf{\Sigma_{XX}})$ et $Z = g(\mathbf{X})$ une fonction d'état limite **continue et dérivable**

$$\boxed{p_f = \Pr(Z \leq 0) = F_Z(0) \approx \Pr\left(U \leq -\frac{\mu_Z}{\sigma_Z}\right) = F_U(-\beta)}$$

où $\beta = \dfrac{\mu_Z}{\sigma_Z}$ est **l'indice de fiabilité** et $U \sim \phi(u)$, $u = \dfrac{z - \mu_Z}{\sigma_Z}$ est une variable aléatoire **normale centrée réduite**.



Comment estimer μ_Z et σ_Z?

Plan du module #5

Introduction
MCFOSM
$\mathbf{X} \to \mathbf{U}$
FOSM
iHL-RF
Exemple #5.2
Résumé

Organisation de la matière

Théorie probabilité
- 0 Révision algèbre & probabilités
- 1 Lois de probabilités
- 3 PDF multivariés

Fiabilité des structures & systèmes
- Introduction
- 2 Formulation fiabilité composantes
- 9 Formulation fiabilité systèmes

Estimation – p_f échantilonnage
- 4 Monte Carlo
- 12 Échantillonnage par importance

Estimation – p_f analytique
- 5 FOSM
- 6 FORM
- 8 SORM

Utilisation des résultats
- 7 Analyse de sensibilité
- 10 Incertitudes aléatoires & épistémiques
- 11 Données empiriques
- 13 Prise de décision
- 14 Métamodèles & modèles empiriques

Plan de la section

MCFOSM

2.1 Definition

2.2 Formulation

2.3 Exemple

Définition

1ère approche **MCFOSM**:

Approximation du premier ordre centrée sur la moyenne et utilisant les moments du second ordre

(Mean-Centered, First-Order, Second-Moment)

Formulation [ ]

Approximation du premier ordre centré sur la moyenne
$\equiv$ linéarisation de $g(\mathbf{X})$ à $\mathbf{M_X}$

$$\begin{aligned} g(\mathbf{X}) &\cong g(\mathbf{M_X}) + \nabla g(\mathbf{M_X})(\mathbf{X} - \mathbf{M_X}) \\ \mu_Z &\cong g(\mathbf{M_X}) \\ \sigma_Z^2 &\cong \nabla g(\mathbf{M_X}) \mathbf{\Sigma_{XX}} \nabla g(\mathbf{M_X})^\intercal \end{aligned}$$

où
$$\nabla g(\mathbf{M_X}) = \left[\frac{\partial g(\mathbf{x})}{\partial x_1}, \cdots, \frac{\partial g(\mathbf{x})}{\partial x_N} \right]_{\mathbf{x}=\mathbf{M_X}}$$

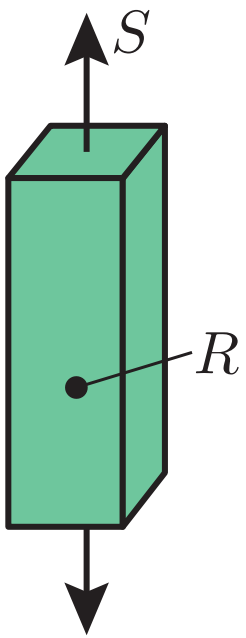
$$\boxed{\beta \cong \beta_{\text{MCFOSM}} = \frac{g(\mathbf{M_X})}{\sqrt{\nabla g(\mathbf{M_X}) \mathbf{\Sigma_{XX}} \nabla g(\mathbf{M_X})^\intercal}}}$$

Limitation: *Dépend du choix de la formulation de $g(\mathbf{X})$*

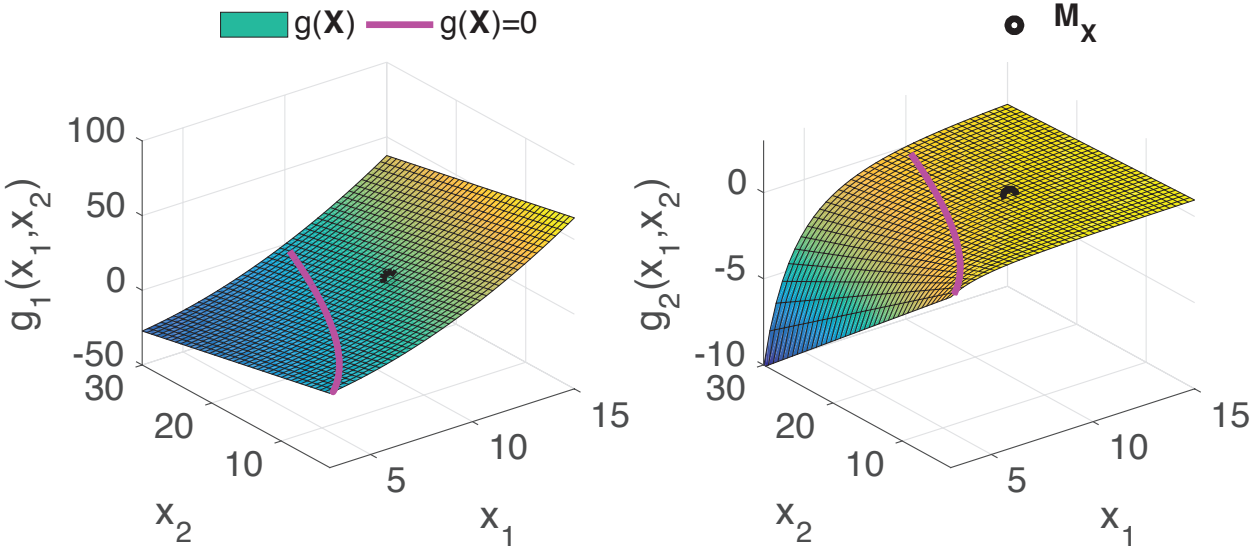
Exemple #5.1

Soit $X_1 = D$ et $X_2 = S$ deux variables aléatoires où:

$$D \left\{ \begin{array}{lcl} \mu_D & = & 10\,\text{mm} \\ \sigma_D & = & 2\,\text{mm} \end{array} \right. , \; S \left\{ \begin{array}{lcl} \mu_S & = & 15\,\text{kN} \\ \sigma_S & = & 5\,\text{kN} \end{array} \right. , \; f_y = 300\,MPa, \; D \perp\!\!\!\perp S$$



$$R(\mathbf{X}) = D^2 f_y, \; \boxed{g_1(\mathbf{X}) = R(\mathbf{X}) - S}, \; \boxed{g_2(\mathbf{X}) = 1 - S/R(\mathbf{X})}$$



Exemple #5.1 (cont.)

Soit $\boxed{g_1(\mathbf{X}) = D^2 f_y - S}$ et $\boxed{g_2(\mathbf{X}) = 1 - \frac{S}{D^2 f_y}}$, où:

$$\begin{array}{rcl} g_1(\mathbf{X}) & \not\equiv & g_2(\mathbf{X}) \\ g_1(\mathbf{X}) = 0 & \equiv & g_2(\mathbf{X}) = 0 \end{array}, \quad \mathbf{M_X} = \left\{ \begin{array}{c} 10 \\ 15 \end{array} \right\}, \quad \mathbf{\Sigma_{XX}} = \begin{bmatrix} 4 & 0 \\ 0 & 25 \end{bmatrix}$$

Solution pour $g_1(\mathbf{X})$

$$\begin{aligned} \mu_Z &\cong g_1(\mathbf{M_X}) = 10^2 f_y - 15 = 15 \quad (f_y = 0.3\, kN/m^2) \\ \nabla g_1(\mathbf{M_X}) &= \left[\frac{\partial g(\mathbf{x})}{\partial x_1}, \frac{\partial g(\mathbf{x})}{\partial x_2} \right]_{\mathbf{x}=\mathbf{M_X}} = [2x_1 f_y, -1]_{\mathbf{x}=\mathbf{M_X}} = [6, -1] \\ \sigma_Z^2 &\cong \nabla g(\mathbf{M_X}) \mathbf{\Sigma_{XX}} \nabla g(\mathbf{M_X})^\intercal \\ &\cong [6, -1] \begin{bmatrix} 4 & 0 \\ 0 & 25 \end{bmatrix} [6, -1]^\intercal = 13^2 \\ \beta &\cong \beta_{\text{MCFOSM}} = \frac{\mu_Z}{\sigma_Z} = \frac{15}{13} \approx 1.15, \quad p_{f,\text{MCFOSM}} = \Phi(-\beta) \approx 0.12 \end{aligned}$$

Exemple #5.1 (cont.)

Soit $\boxed{g_1(\mathbf{X}) = D^2 f_y - S}$ et $\boxed{g_2(\mathbf{X}) = 1 - \frac{S}{D^2 f_y}}$, où:

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Solution pour $g_2(\mathbf{X})$

$$\begin{aligned} \mu_Z &\cong g_2(\mathbf{M_X}) = 1 - 15/(10^2 f_y) = 0.5 \\ \nabla g_2(\mathbf{M_X}) &= \left[2x_2/(f_y x_1^3), -1/(fyx_1^2)\right]_{\mathbf{x}=\mathbf{M_X}} = [0.10, -0.03] \\ \sigma_Z^2 &\cong \nabla g(\mathbf{M_X}) \mathbf{\Sigma_{XX}} \nabla g(\mathbf{M_X})^\intercal \\ &\cong [0.10, -0.03] \begin{bmatrix} 4 & 0 \\ 0 & 25 \end{bmatrix} [0.10, -0.03]^\intercal = 0.21^2 \\ \beta &\cong \beta_{\text{MCFOSM}} = \frac{\mu_Z}{\sigma_Z} = \frac{0.5}{0.26} \approx 1.92 \gg 1.15 \quad ⚠ \end{aligned}$$

Exemple #5.1 (cont.) [.m]

Soit $\boxed{g_1(\mathbf{X}) = D^2 f_y - S}$ et $\boxed{g_2(\mathbf{X}) = 1 - \frac{S}{D^2 f_y}}$, où:

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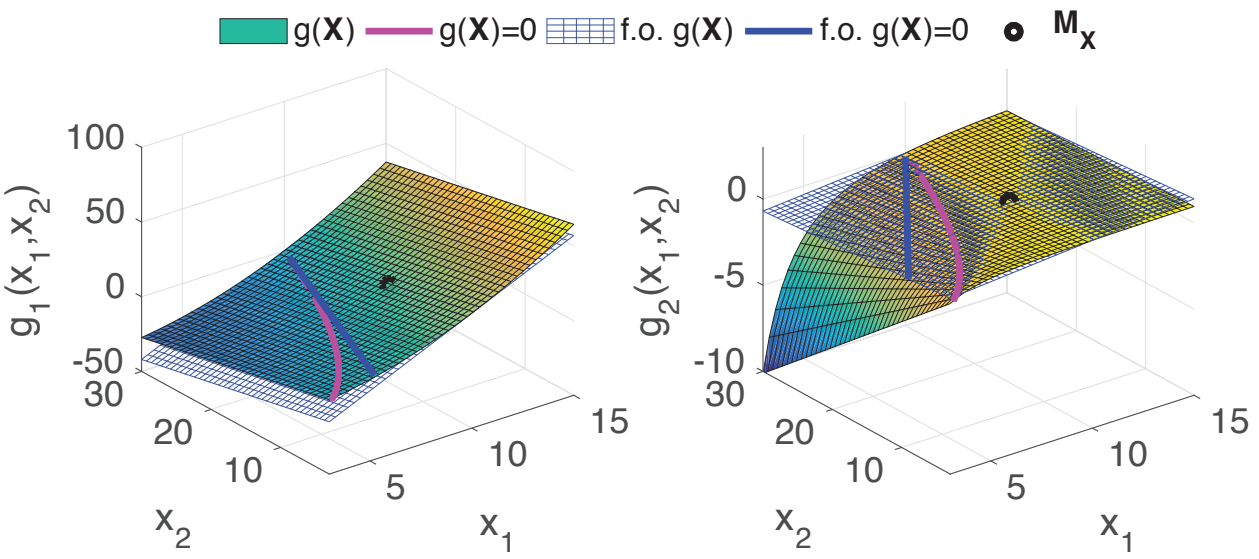


$p_{f,\text{true}} = \mathbf{0.08}$, g_1 : $p_{f,\text{MCFOSM}} \approx 0.12$ ⚠, g_2 : $p_{f,\text{MCFOSM}} \approx 0.027$ ⚠

Exemple #5.1 (cont.) [.m]

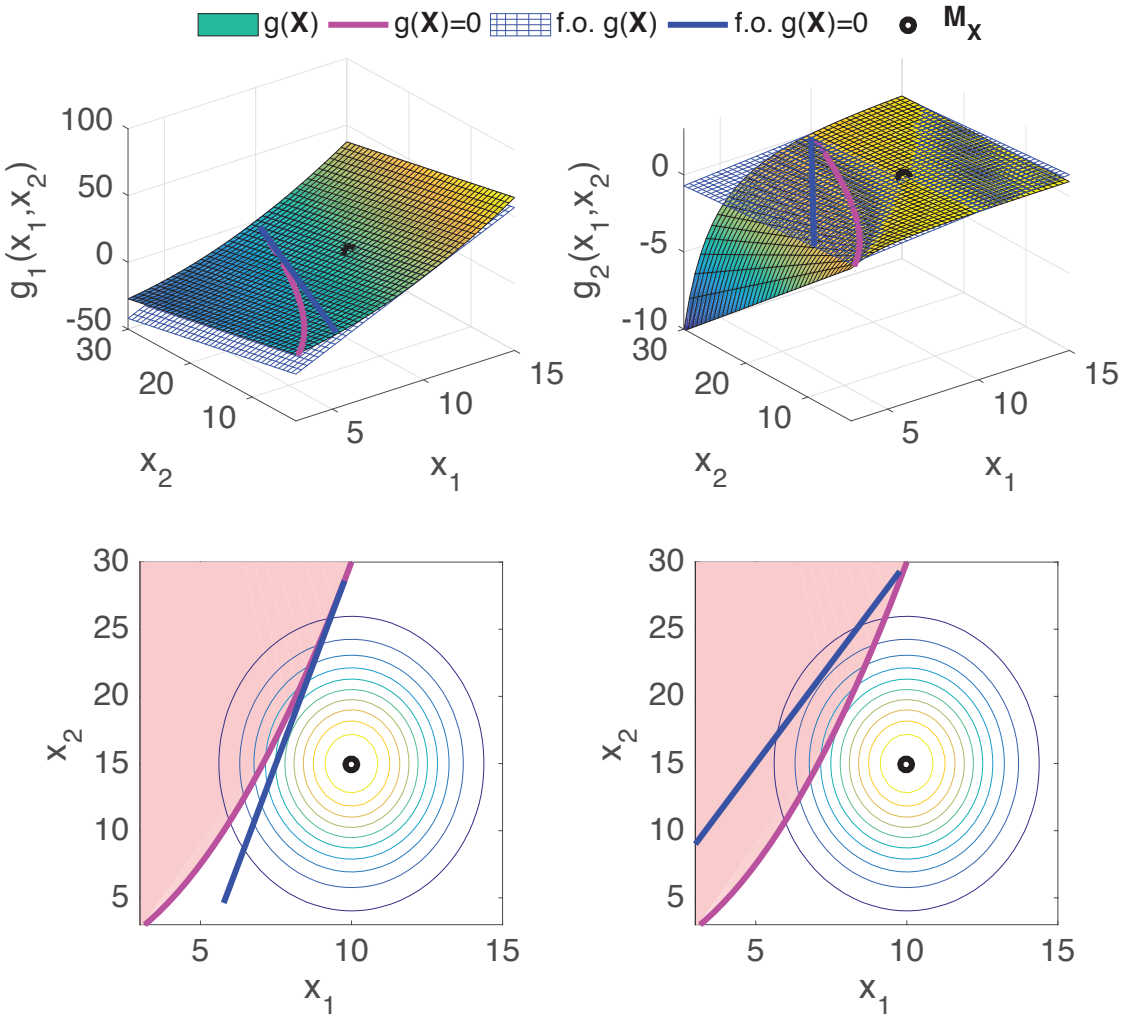
$$\boxed{g_1(\mathbf{X}) = D^2 f_y - S}$$

$$\boxed{g_2(\mathbf{X}) = 1 - \frac{S}{D^2 f_y}}$$

$$p_f = \mathbf{0.08}$$

$$g_1 : p_{f,\text{MCFOSM}} \approx 0.12$$

$$g_2 : p_{f,\text{MCFOSM}} \approx 0.027$$



Plan de la section

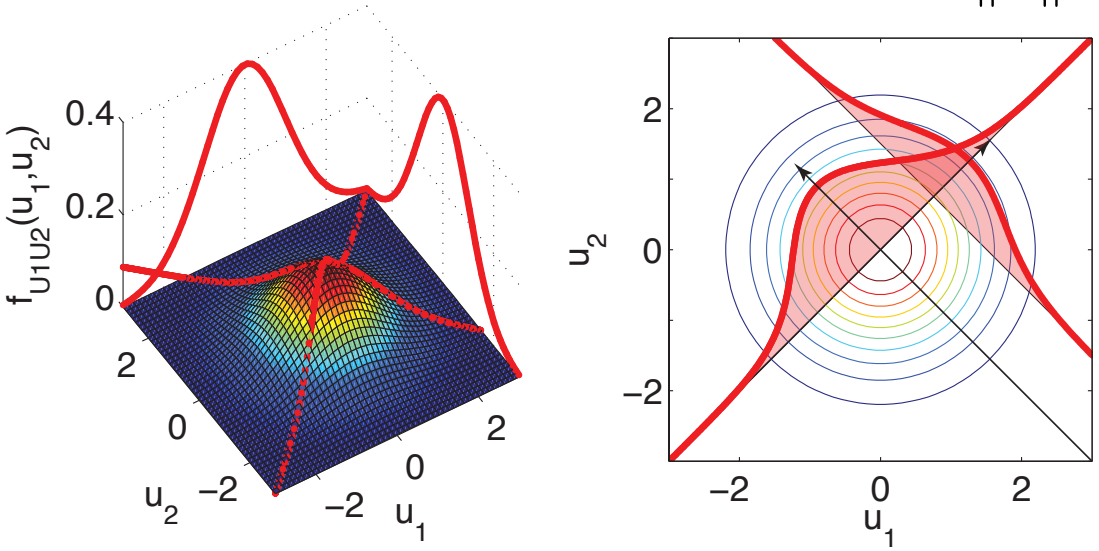
$\mathbf{X} \to \mathbf{U}$
3.1 Variables normales centrées réduites
3.2 Transformation $\mathbf{X} \to \mathbf{U}$
3.3 Interprétation de β dans l'espace NCR
3.4 Probabilité de défaillance

Variables normales centrées réduites

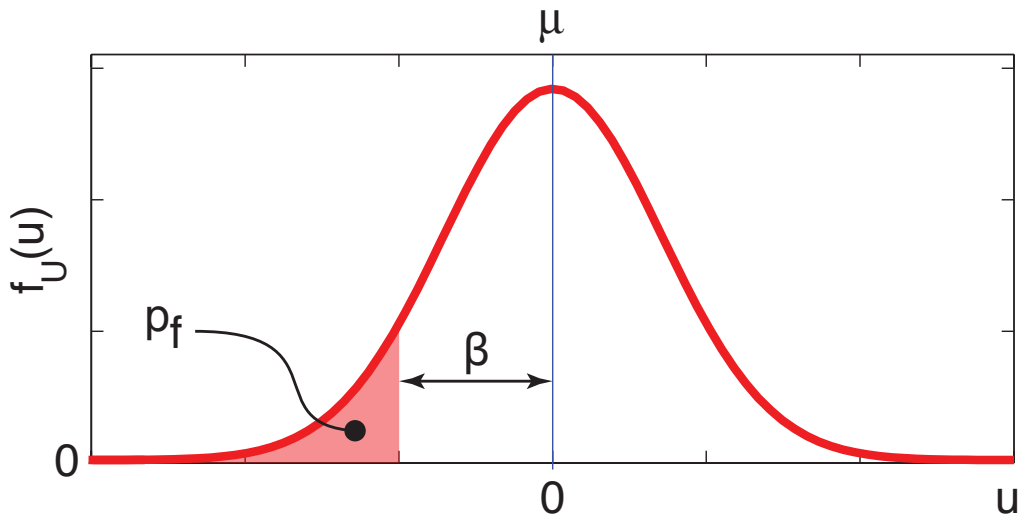
Soit $\mathbf{U} = \{U_1, \cdots, U_n\}$ un ensemble de variables aléatoires normales centrée réduites tel que

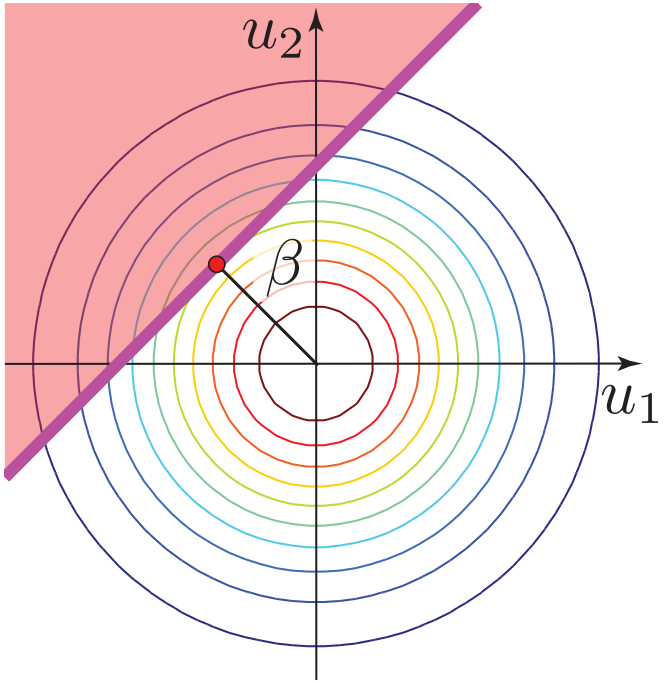
$$\mathbf{U} \sim \phi_n(\mathbf{u}) = \frac{1}{(2\pi)^{n/2}} \exp\left(-\frac{\|\mathbf{u}\|^2}{2}\right)$$

- Les contours de $\phi_n(\mathbf{u})$ sont sphériques et centrés à l'origine
- La probabilité dans toutes les directions radiales diminue exponentiellement, i.e. avec un incrément de $\|\mathbf{u}\|^2$



Variables normales centrées réduites – p_f





p_f est complètement définie par la distance entre l'origine et la surface d'état limite (⚠ fct. linéaire).

Transformation $\mathbf{X} \to \mathbf{U}$

Soit $\mathbf{X} \sim (\mathbf{M_X}, \mathbf{\Sigma_{XX}})$ et soit une fonction d'état limite $g(\mathbf{X})$.

Transformation: Si $\mathbf{X} \sim \mathcal{N}(\mathbf{M_X}, \mathbf{\Sigma_{XX}})$, on peut transformer $\mathbf{X}$ en $\mathbf{U} \sim \phi_n(\mathbf{u})$ tel que

$$\begin{aligned} \mathbf{U} &= \mathbf{L}^{-1}\mathbf{D_X}^{-1}(\mathbf{X} - \mathbf{M_X}) \\ \mathbf{X} &= \mathbf{M_X} + \mathbf{D_X}\mathbf{L}\mathbf{U} \end{aligned}$$

où $\mathbf{\Sigma_{XX}} = \mathbf{D_X}\mathbf{R_{XX}}\mathbf{D_X}$, $\mathbf{D_X} = \text{diag}[\sigma_{x_i}]$, $\mathbf{R_{XX}} = \mathbf{L}\mathbf{L}^\intercal$, et $\mathbf{L} = \text{Chol}(\mathbf{R_{XX}})$

$$\boxed{g(\mathbf{X}) = g(\mathbf{M_X} + \mathbf{D_X}\mathbf{L}\mathbf{U}) \equiv G(\mathbf{U})}$$

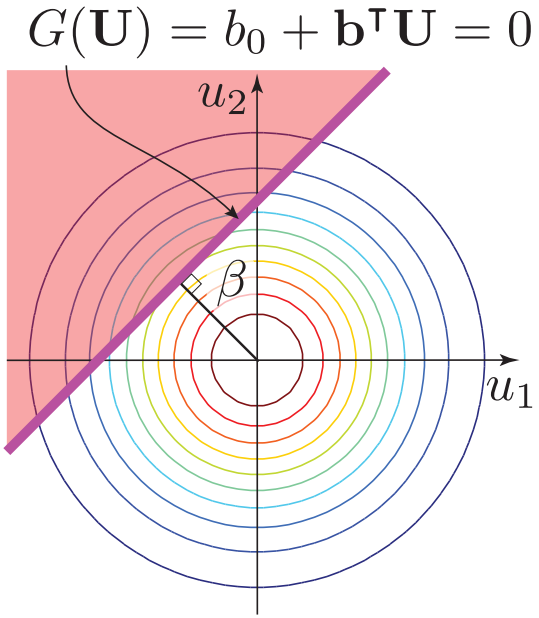
L: Matrice triangulaire inférieure

Interpretation de β dans l'espace NCR [ ]

Soit $G(\mathbf{X}) = a_0 + \sum_{i=1}^{n} a_i X_i = a_0 + \mathbf{a}^\intercal \mathbf{X}, \quad \mathbf{a} = [a_1, a_2, \cdots, a_n]^\intercal$

$$\begin{aligned} \mathbf{U} &= \mathbf{L}^{-1}\mathbf{D}_{\mathbf{X}}^{-1}(\mathbf{X} - \mathbf{M}_{\mathbf{X}}) \\ \mathbf{X} &= \mathbf{M}_{\mathbf{X}} + \mathbf{D}_{\mathbf{X}}\mathbf{L}\mathbf{U} \end{aligned}$$

$$\begin{aligned} G(\mathbf{U}) \equiv g(\mathbf{X}) &= a_0 + \mathbf{a}^\intercal(\mathbf{M}_{\mathbf{X}} + \mathbf{D}_{\mathbf{X}}\mathbf{L}\mathbf{U}) \\ &= \underbrace{a_0 + \mathbf{a}^\intercal \mathbf{M}_{\mathbf{X}}}_{b_0} + \underbrace{\mathbf{a}^\intercal \mathbf{D}_{\mathbf{X}}\mathbf{L}}_{\mathbf{b}^\intercal}\mathbf{U} \\ &= b_0 + \mathbf{b}^\intercal \mathbf{U} \end{aligned}$$



Comme $\mathbf{U} \sim \mathcal{N}(\mathbf{0}, \mathbf{I})$,

$$\begin{aligned} \mu_{G(\mathbf{U})} &= \mathbb{E}[G(\mathbf{U})] &&= b_0 \\ \sigma^2_{G(\mathbf{U})} &= \mathrm{Var}[G(\mathbf{U})] &&= \mathbf{b}^\intercal \mathbf{I} \mathbf{b} = \|\mathbf{b}\|^2 \end{aligned}$$

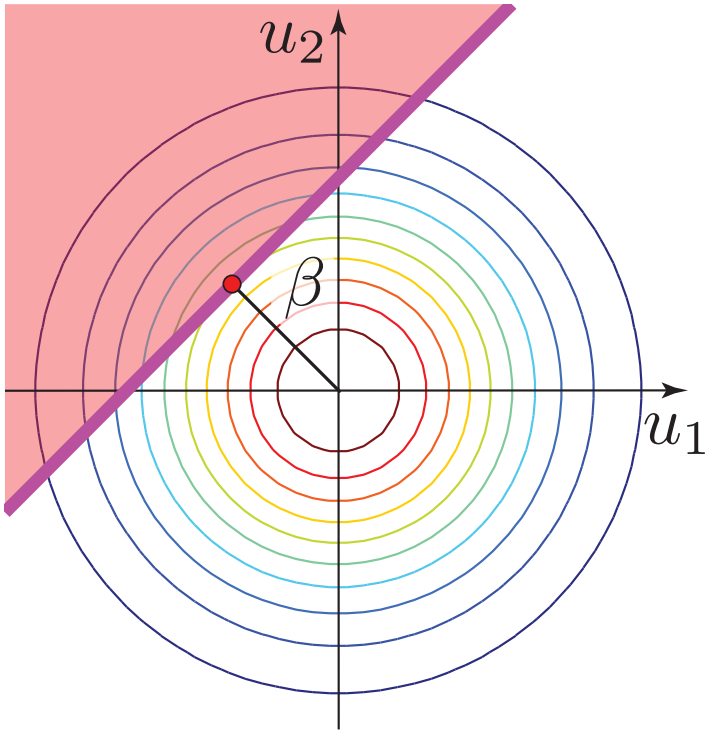
$$\boxed{\beta = \frac{\mu_{G(\mathbf{U})}}{\sigma_{G(\mathbf{U})}} = \frac{b_0}{\|\mathbf{b}\|}}$$

β: distance entre l'origine et le plan $G(\mathbf{U}) = 0$

$\beta \to p_f$

La probabilité du demi-espace $\Omega = \{\mathbf{u} : G(\mathbf{u}) \leq 0\}$ borné par un plan est **définie par la distance entre l'origine et le plan**.

$$
\begin{aligned}
\Pr(\mathbf{U} \in \Omega) &= \int_{\Omega} \phi_n(\mathbf{u}) d\mathbf{u} \\
&= \Phi(-\beta) \\
&= \int_{\mathbf{x}:g(\mathbf{x})<0} f_{\mathbf{X}} d\mathbf{x} \\
&= p_f
\end{aligned}
$$



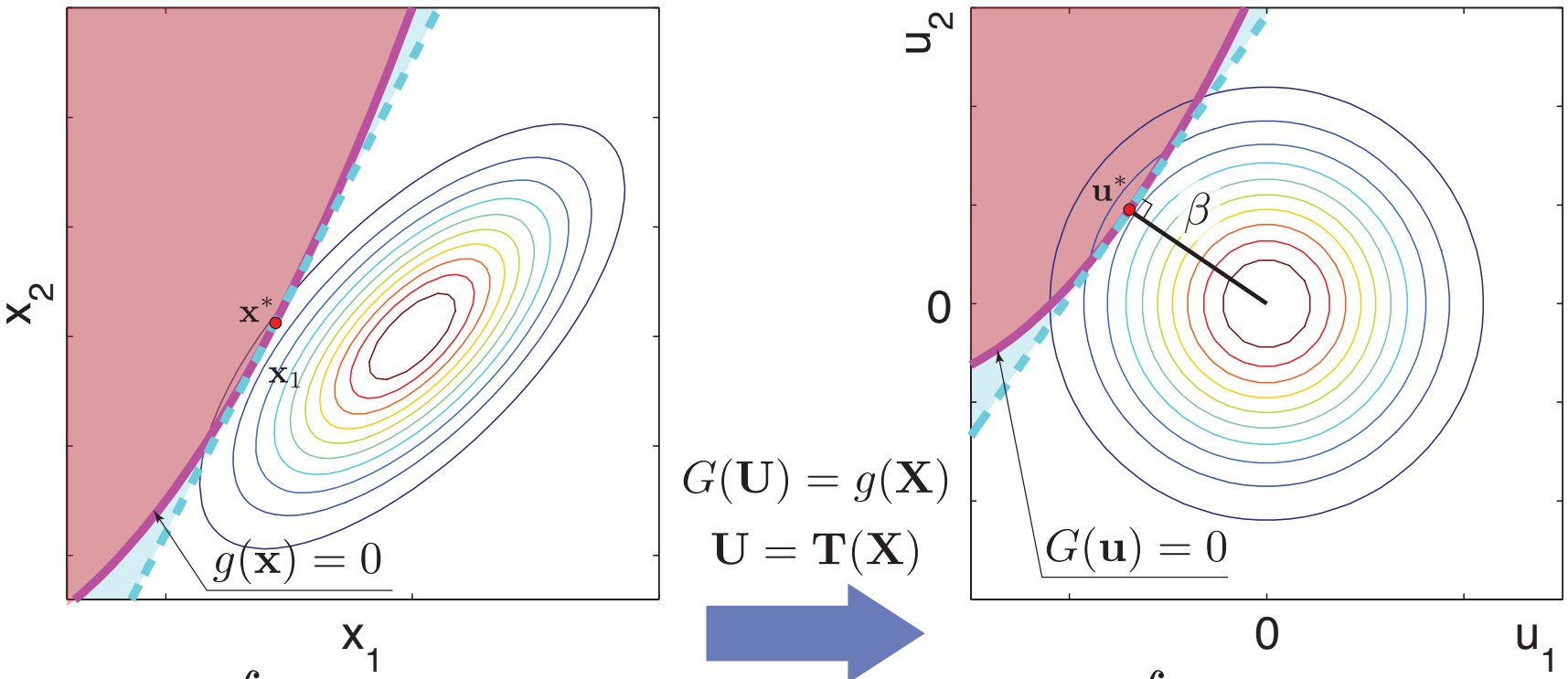
Plan de la section

FOSM

4.1 Définition

4.2 Formulation

FOSM: First Order Second Moments



$$p_f = \int_{\mathbf{x}:g(\mathbf{x})<0} f_{\mathbf{X}}(\mathbf{x})d\mathbf{x}$$

$$p_f = \int_{\mathbf{u}:G(\mathbf{u})<0} f_{\mathbf{u}}(\mathbf{u})d\mathbf{u}$$

FOSM: First Order Second Moments

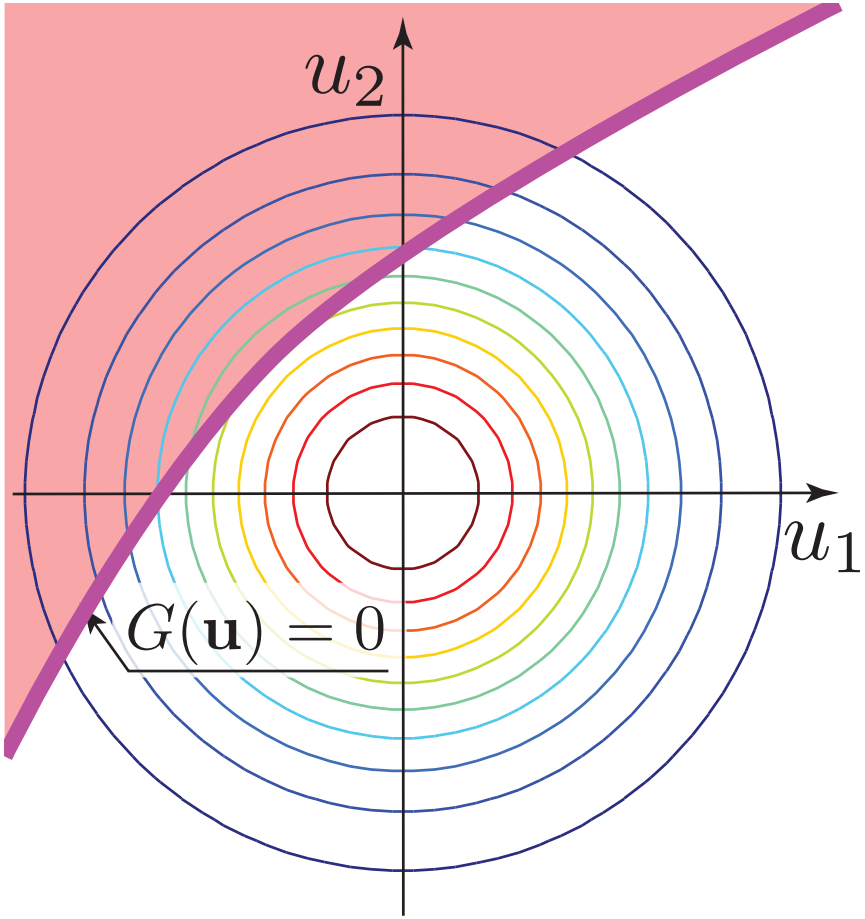
$$\mathbf{U} = \mathbf{L}^{-1}\mathbf{D}_{\mathbf{X}}^{-1}(\mathbf{X} - \mathbf{M}_{\mathbf{X}})$$

$$\overbrace{\mathbf{M}_{\mathbf{X}}}^{\text{espace } \mathbf{x}} \to \overbrace{\mathbf{0}}^{\text{espace } \mathbf{u}}$$

MCFOSM: linéarisation de $G(\mathbf{u})$ à $\mathbf{u} = \mathbf{0}$

FOSM: linéarisation $G(\mathbf{u})$ à $\mathbf{u}^*$

> $\mathbf{u}^*$: *Point de défaillance le plus probable*



FOSM: First Order Second Moments

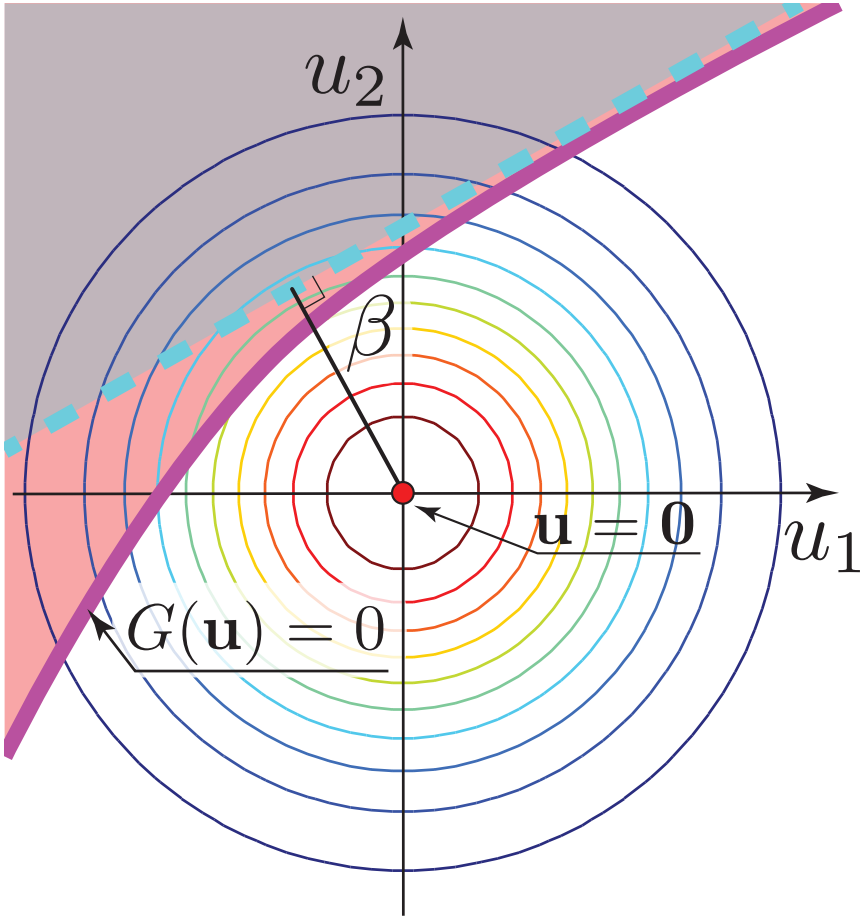
$$\mathbf{U} = \mathbf{L}^{-1}\mathbf{D}_{\mathbf{X}}^{-1}(\mathbf{X} - \mathbf{M}_{\mathbf{X}})$$

$$\overbrace{\mathbf{M}_{\mathbf{X}}}^{\text{espace } \mathbf{x}} \to \overbrace{\mathbf{0}}^{\text{espace } \mathbf{u}}$$

MCFOSM: linéarisation de $G(\mathbf{u})$ à $\mathbf{u} = \mathbf{0}$

FOSM: linéarisation $G(\mathbf{u})$ à $\mathbf{u}^*$

> $\mathbf{u}^*$: *Point de défaillance le plus probable*



FOSM: First Order Second Moments

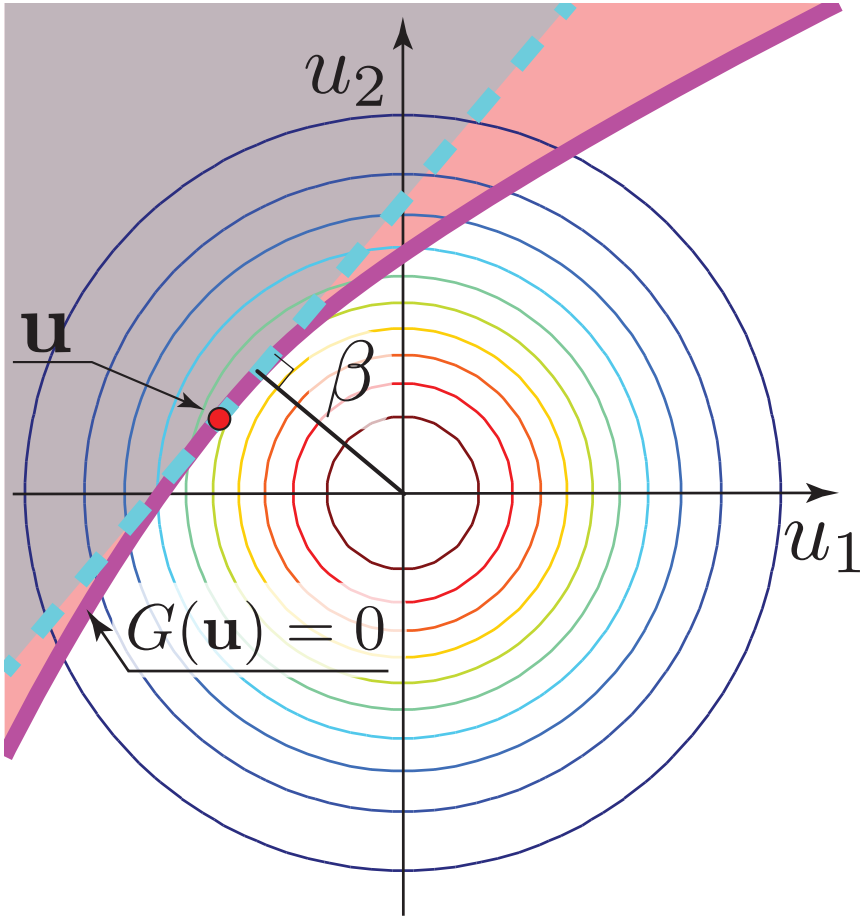
$$\mathbf{U} = \mathbf{L}^{-1}\mathbf{D}_{\mathbf{X}}^{-1}(\mathbf{X} - \mathbf{M}_{\mathbf{X}})$$

$$\overbrace{\mathbf{M}_{\mathbf{X}}}^{\text{espace } \mathbf{x}} \to \overbrace{\mathbf{0}}^{\text{espace } \mathbf{u}}$$

MCFOSM: linéarisation de $G(\mathbf{u})$ à $\mathbf{u} = \mathbf{0}$

FOSM: linéarisation $G(\mathbf{u})$ à $\mathbf{u}^*$

> $\mathbf{u}^*$: *Point de défaillance le plus probable*



FOSM: First Order Second Moments

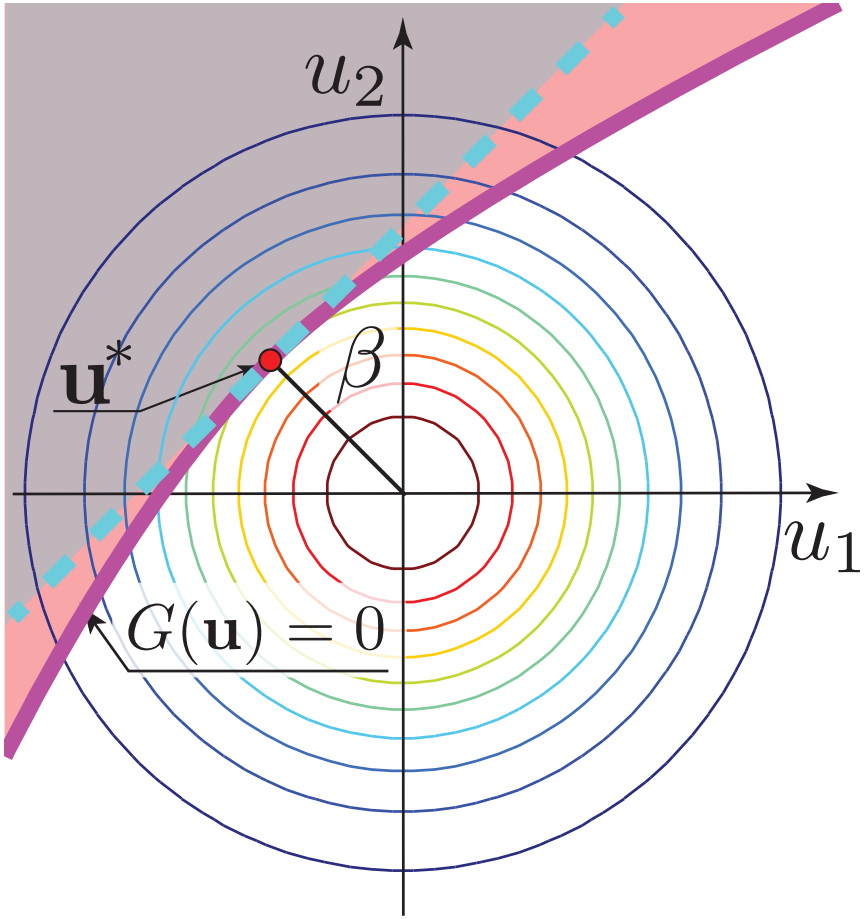
$$\mathbf{U} = \mathbf{L}^{-1}\mathbf{D}_{\mathbf{X}}^{-1}(\mathbf{X} - \mathbf{M}_{\mathbf{X}})$$

$$\overbrace{\mathbf{M}_{\mathbf{X}}}^{\text{espace } \mathbf{x}} \to \overbrace{\mathbf{0}}^{\text{espace } \mathbf{u}}$$

MCFOSM: linéarisation de $G(\mathbf{u})$ à $\mathbf{u} = \mathbf{0}$

FOSM: linéarisation $G(\mathbf{u})$ à $\mathbf{u}^*$

> $\mathbf{u}^*$: *Point de défaillance le plus probable*



Définition

$2^{\text{ème}}$ approche **FOSM**:

Approximation du premier ordre centré sur le point de défaillance le plus probable (PDPP) et utilisant les moments du second ordre

Avantage: *Invariable face au choix de la formulation de* $g(\mathbf{X})$

Comment?

1. Transformation de $g(\mathbf{X})$ dans l'**espace normal centré réduit**
2. Linéarisation de $G(\mathbf{U})$ centrée sur le **point de défaillance le plus probable** $\mathbf{u}^*$

Formulation FOSM

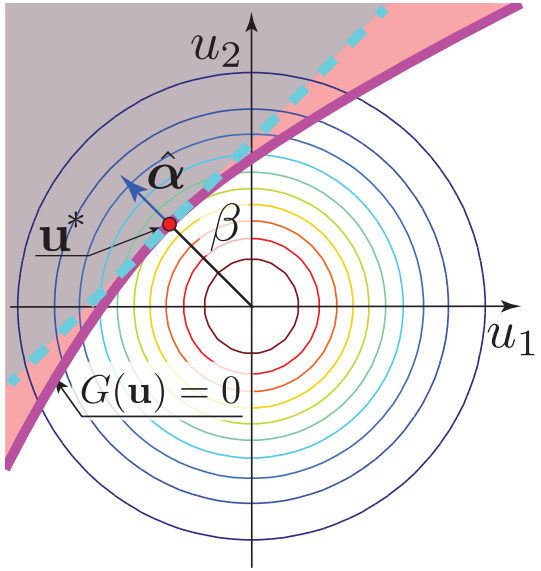
Linérarisation: On linéarise $G(\mathbf{U})$ au point $\mathbf{u}^*$ dans l'espace NCR

$$\begin{aligned} G(\mathbf{U}) &\cong \overbrace{G(\mathbf{u}^*)}^{\equiv 0} + \nabla G(\mathbf{u}^*)(\mathbf{U} - \mathbf{u}^*) \\ &= \nabla G(\mathbf{u}^*)\mathbf{U} - \nabla G(\mathbf{u}^*)\mathbf{u}^* \end{aligned}$$

$$\boxed{\nabla G(\mathbf{u}) = \nabla g(\mathbf{x})\mathbf{J}_{\mathbf{x},\mathbf{u}}, \;\; \mathbf{J}_{\mathbf{x},\mathbf{u}} = \mathbf{D}_{\mathbf{X}}\mathbf{L}}$$

$$\begin{aligned} \mu_{G(\mathbf{U})} &= -\nabla G(\mathbf{u}^*)\mathbf{u}^* \\ \sigma^2_{G(\mathbf{U})} &= \nabla G(\mathbf{u}^*)^\intercal \mathbf{\Sigma}_{\mathbf{UU}} \nabla G(\mathbf{u}^*) \\ &= \nabla G(\mathbf{u}^*)^\intercal \mathbf{I} \nabla G(\mathbf{u}^*) = \|\nabla G(\mathbf{u}^*)\|^2 \end{aligned}$$

$$\boxed{\beta \cong \beta_{\text{FOSM}} = \frac{\mu_{G(\mathbf{U})}}{\sigma_{G(\mathbf{U})}} = \underbrace{-\frac{\nabla G(\mathbf{u}^*)}{\|\nabla G(\mathbf{u}^*)\|}}_{\hat{\boldsymbol{\alpha}}} \mathbf{u}^* = \hat{\boldsymbol{\alpha}}\mathbf{u}^*}$$



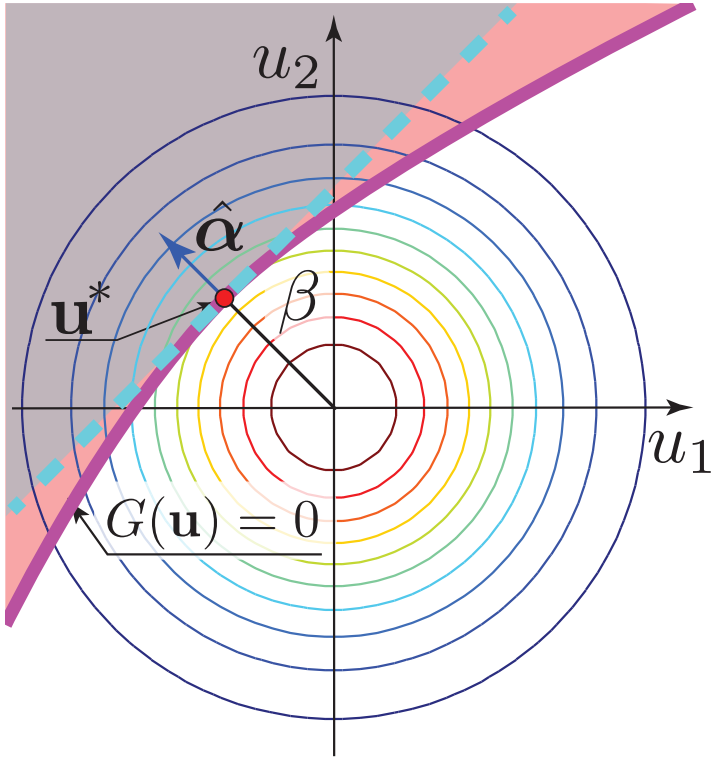
Formulation FOSM (cont.)

$\hat{\boldsymbol{\alpha}}$: Vecteur gradient négatif normalisé, i.e. vecteur unitaire perpendiculaire à $G(\mathbf{u}^*) = 0$ et pointant vers le domaine de défaillance

$$\boxed{\hat{\boldsymbol{\alpha}} = -\frac{\nabla G(\mathbf{u}^*)}{\|\nabla G(\mathbf{u}^*)\|}}$$

$\mathbf{u}^*$: *Point de conception*, i.e., point de défaillance le plus probable

$$\boxed{\mathbf{u}^* = \beta \hat{\boldsymbol{\alpha}}^\mathsf{T} = \arg\min_{\mathbf{u}} \{\|\mathbf{u}\| \,|\, G(\mathbf{u}) = 0\}}$$



Prochaine étape: *Comment trouver* $\mathbf{u}^*$*?*

Comment trouver le point de conception $\mathbf{u}^*$?

⚠ Il n'y a pas de solution analytique pour trouver $\mathbf{u}^*$

Algorithme iHL-RF: **i***mproved* **H***asofer-***L***ind*, **R***ackwitz-***F***iessler*
iHL-RF est un algorithme itératif basé sur la méthode de Newton-Raphson

Algorithme *iHL-RF* – *point de conception* $\mathbf{u}^*$

À l'itération i, (pour $i = 0$, $\mathbf{u}_i = \mu_{\mathbf{u}} \equiv \mathbf{0}$)

$$\begin{aligned} G(\mathbf{U}) &\cong \overbrace{G(\mathbf{u}_i)}^{\neq \mathbf{0}} + \nabla G(\mathbf{u}_i)(\mathbf{U} - \mathbf{u}_i) \\ &\cong \underbrace{G(\mathbf{u}_i) - \nabla G(\mathbf{u}_i)\mathbf{u}_i}_{\mu_{G(\mathbf{U})}} + \nabla G(\mathbf{u}_i)\mathbf{U} \end{aligned}$$

$$\boxed{\beta_i \cong \frac{\mu_{G(\mathbf{U})}}{\sigma_{G(\mathbf{U})}} = \frac{G(\mathbf{u}_i) - \nabla G(\mathbf{u}_i)\mathbf{u}_i}{\|\nabla G(\mathbf{u}_i)\|} = \frac{G(\mathbf{u}_i)}{\|\nabla G(\mathbf{u}_i)\|} + \hat{\boldsymbol{\alpha}}_i \mathbf{u}_i}$$

$$\boxed{\mathbf{u}_{i+1} = \beta_i \hat{\boldsymbol{\alpha}}_i^{\mathsf{T}} = \left[\frac{G(\mathbf{u}_i)}{\|\nabla G(\mathbf{u}_i)\|} + \hat{\boldsymbol{\alpha}}_i \mathbf{u}_i\right] \hat{\boldsymbol{\alpha}}_i^{\mathsf{T}}}$$

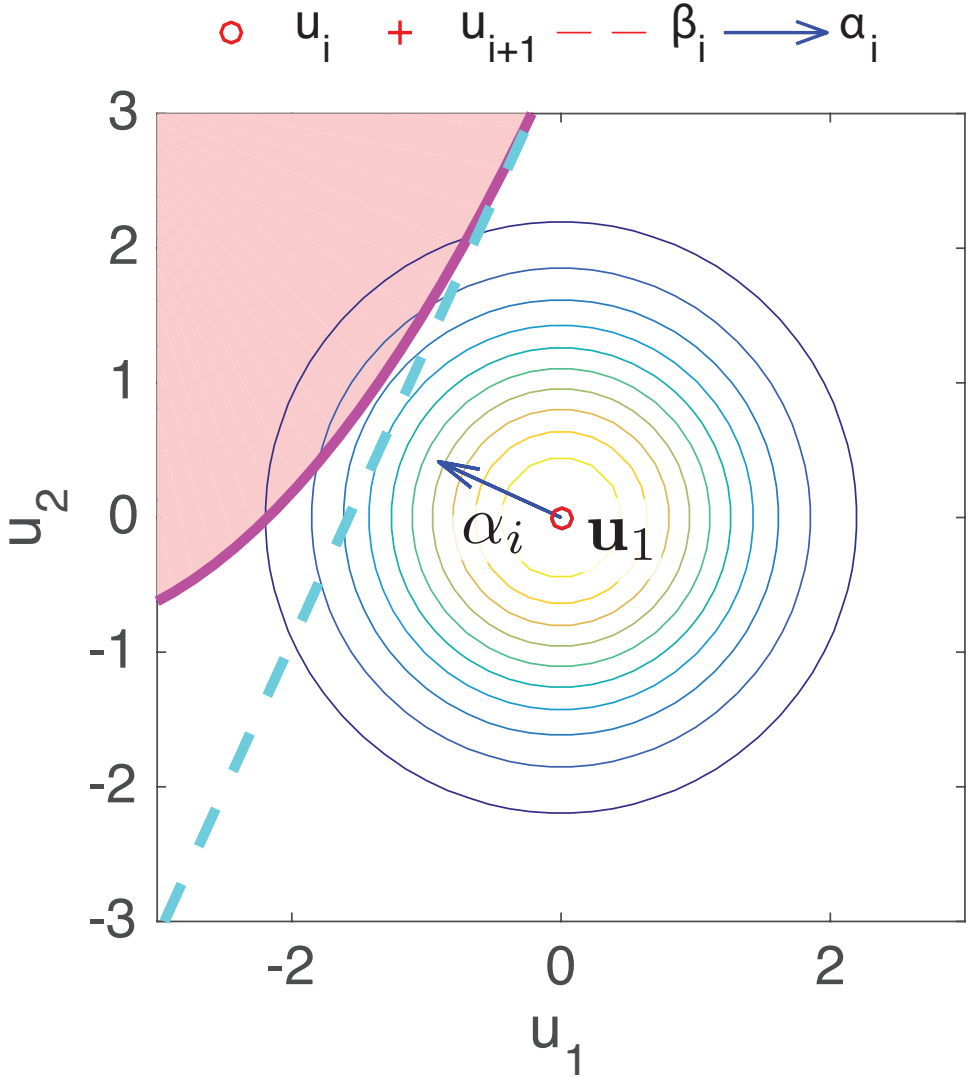
Exemple : Algorithme *iHL-RF*

$$\boxed{\hat{\boldsymbol{\alpha}}_i = -\frac{\nabla G(\mathbf{u}_i)}{\|\nabla G(\mathbf{u}_i)\|}}$$

$$\boxed{\beta_i \cong \frac{G(\mathbf{u}_i)}{\|\nabla G(\mathbf{u}_i)\|} + \hat{\boldsymbol{\alpha}}_i \mathbf{u}_i}$$

$$\boxed{\mathbf{u}_{i+1} = \beta_i \hat{\boldsymbol{\alpha}}_i^{\mathsf{T}}}$$

$$i = 1, \ \mathbf{u}_1 = \mu_{\mathbf{u}} \equiv \mathbf{0}$$



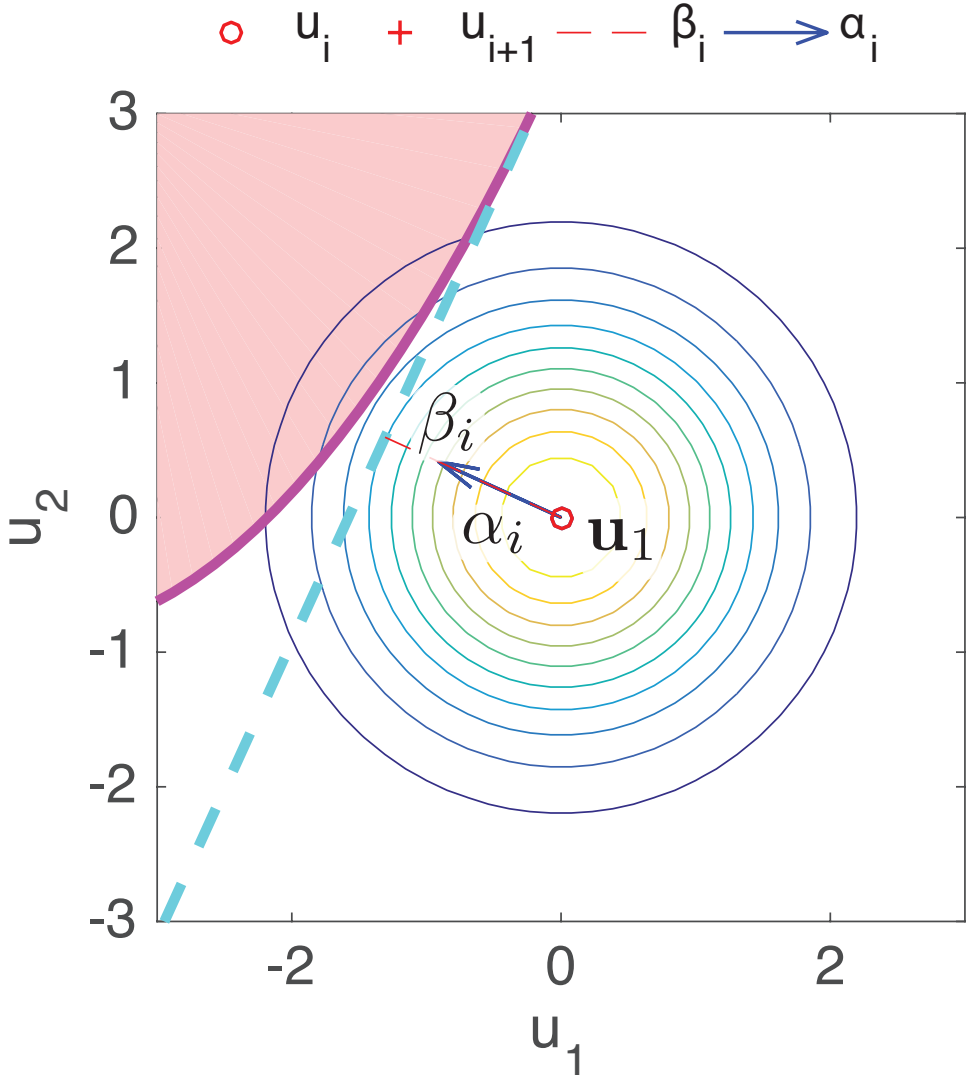
Exemple : Algorithme *iHL-RF*

$$\boxed{\hat{\boldsymbol{\alpha}}_i = -\frac{\nabla G(\mathbf{u}_i)}{\|\nabla G(\mathbf{u}_i)\|}}$$

$$\boxed{\beta_i \cong \frac{G(\mathbf{u}_i)}{\|\nabla G(\mathbf{u}_i)\|} + \hat{\boldsymbol{\alpha}}_i \mathbf{u}_i}$$

$$\boxed{\mathbf{u}_{i+1} = \beta_i \hat{\boldsymbol{\alpha}}_i^{\mathsf{T}}}$$

$$i = 1, \ \mathbf{u}_1 = \mu_{\mathbf{u}} \equiv \mathbf{0}$$



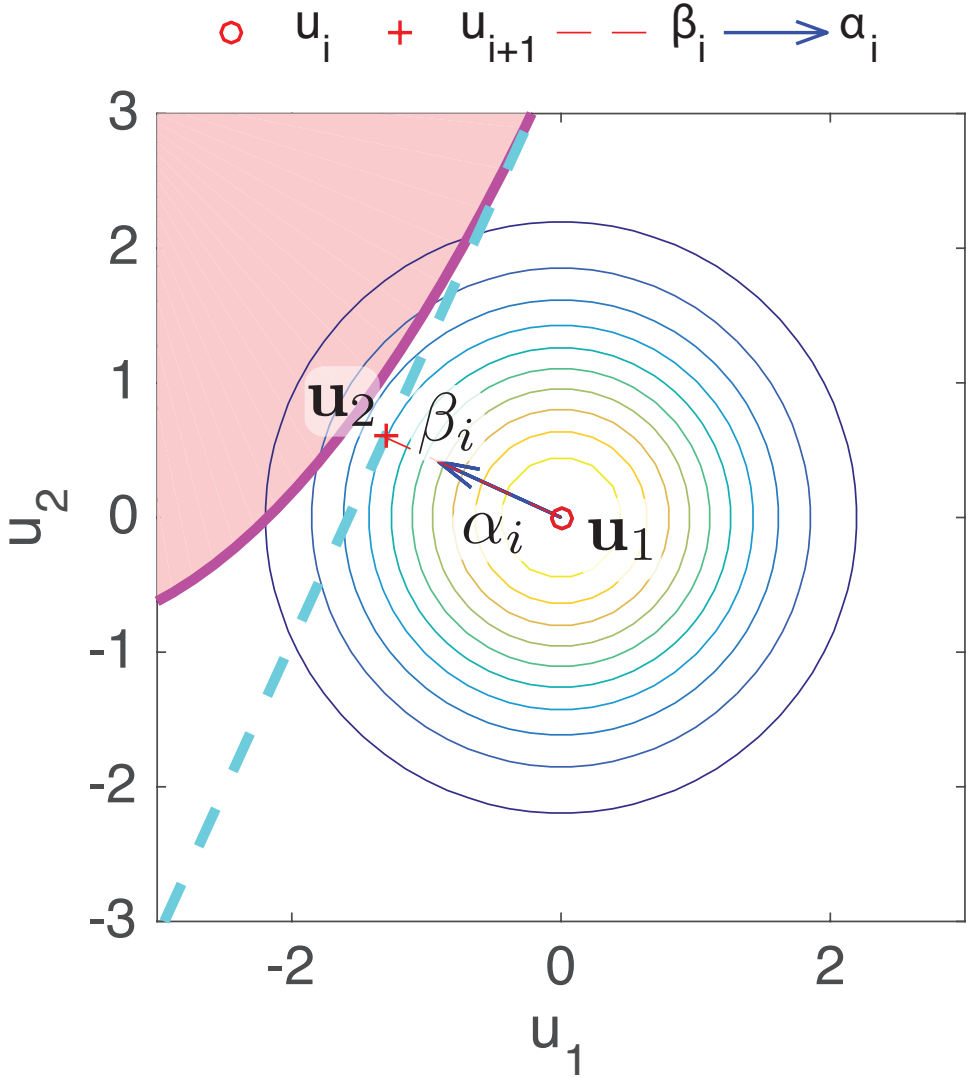
Exemple : Algorithme *iHL-RF*

$$\boxed{\hat{\boldsymbol{\alpha}}_i = -\frac{\nabla G(\mathbf{u}_i)}{\|\nabla G(\mathbf{u}_i)\|}}$$

$$\boxed{\beta_i \cong \frac{G(\mathbf{u}_i)}{\|\nabla G(\mathbf{u}_i)\|} + \hat{\boldsymbol{\alpha}}_i \mathbf{u}_i}$$

$$\boxed{\mathbf{u}_{i+1} = \beta_i \hat{\boldsymbol{\alpha}}_i^{\mathsf{T}}}$$

$$i = 1, \ \mathbf{u}_1 = \mu_{\mathbf{u}} \equiv \mathbf{0}$$



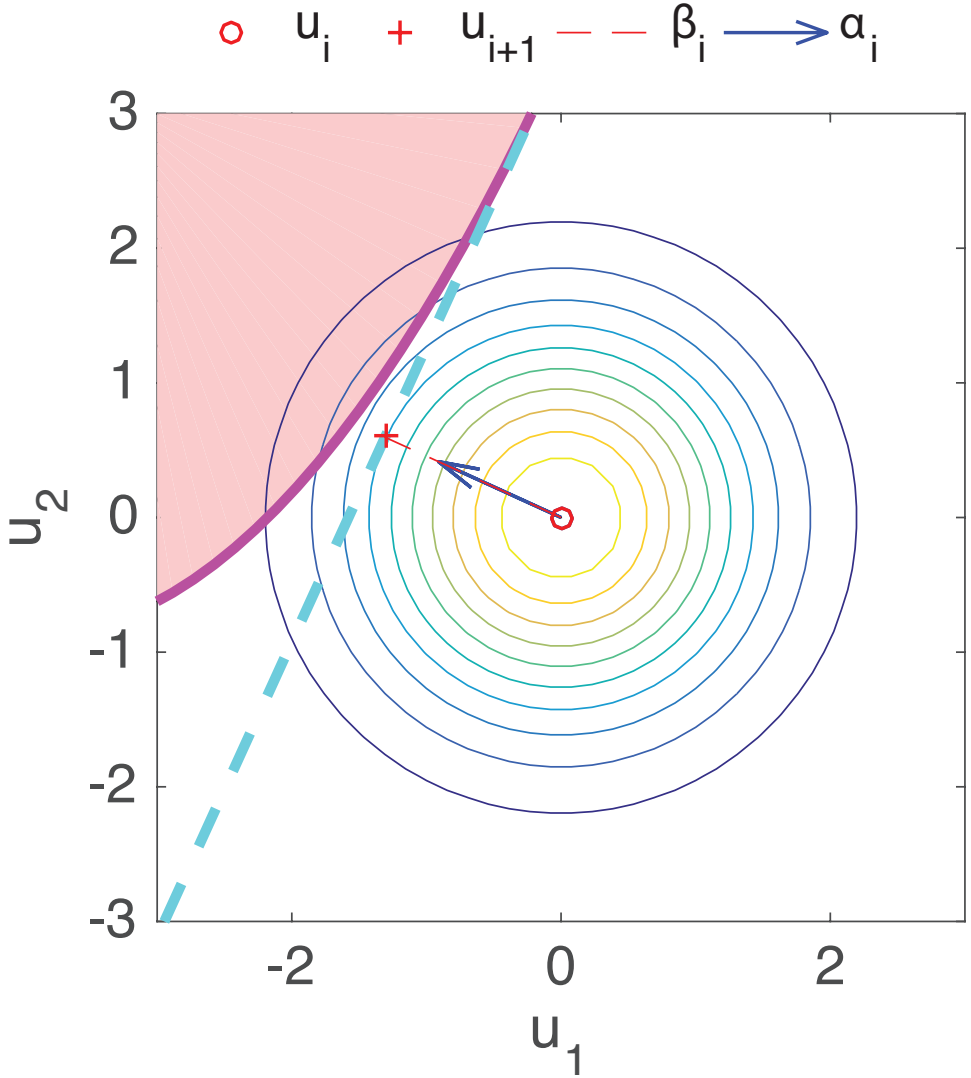
Exemple : Algorithme *iHL-RF*

$$\boxed{\hat{\boldsymbol{\alpha}}_i = -\frac{\nabla G(\mathbf{u}_i)}{\|\nabla G(\mathbf{u}_i)\|}}$$

$$\boxed{\beta_i \cong \frac{G(\mathbf{u}_i)}{\|\nabla G(\mathbf{u}_i)\|} + \hat{\boldsymbol{\alpha}}_i \mathbf{u}_i}$$

$$\boxed{\mathbf{u}_{i+1} = \beta_i \hat{\boldsymbol{\alpha}}_i^{\mathsf{T}}}$$

$$i = 1, \ \mathbf{u}_1 = \mu_{\mathbf{u}} \equiv \mathbf{0}$$



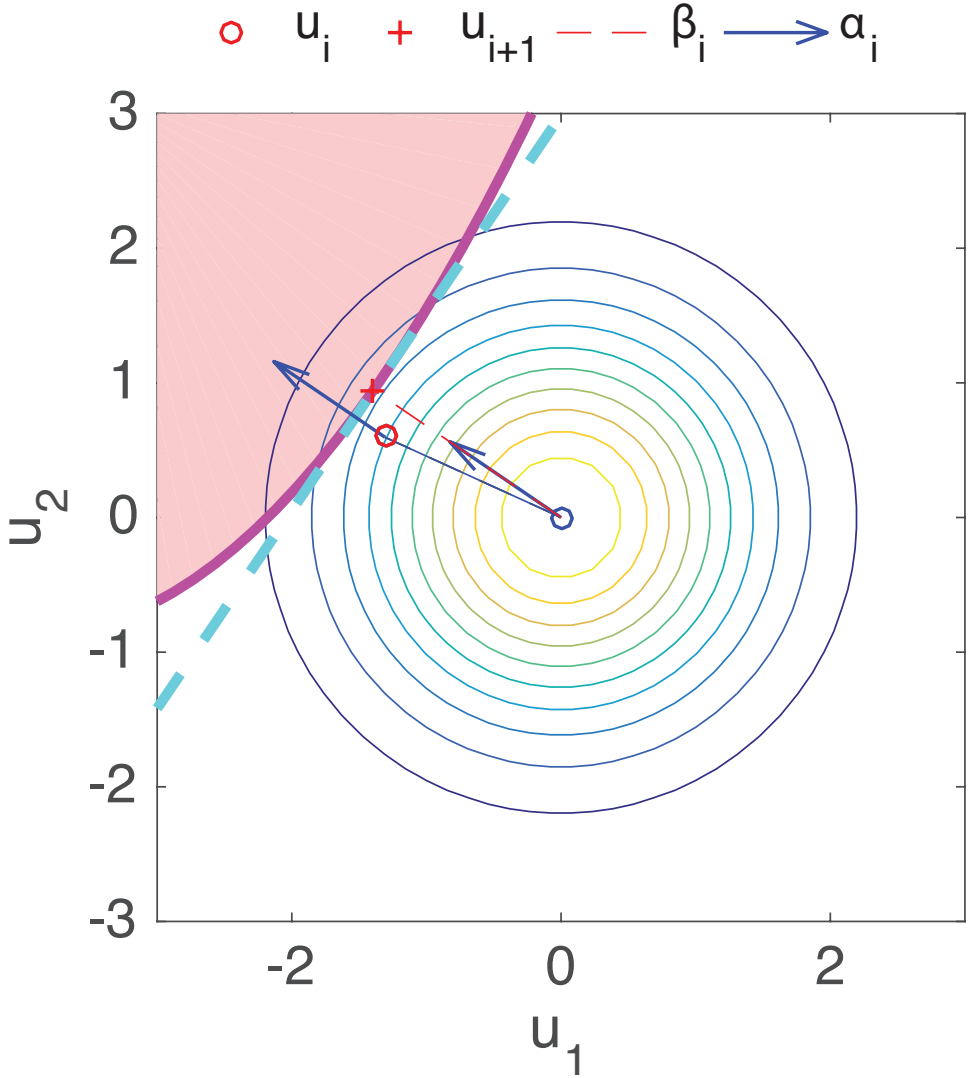
Exemple : Algorithme *iHL-RF*

$$\boxed{\hat{\boldsymbol{\alpha}}_i = -\frac{\nabla G(\mathbf{u}_i)}{\|\nabla G(\mathbf{u}_i)\|}}$$

$$\boxed{\beta_i \cong \frac{G(\mathbf{u}_i)}{\|\nabla G(\mathbf{u}_i)\|} + \hat{\boldsymbol{\alpha}}_i \mathbf{u}_i}$$

$$\boxed{\mathbf{u}_{i+1} = \beta_i \hat{\boldsymbol{\alpha}}_i^{\mathsf{T}}}$$

$i = 2$



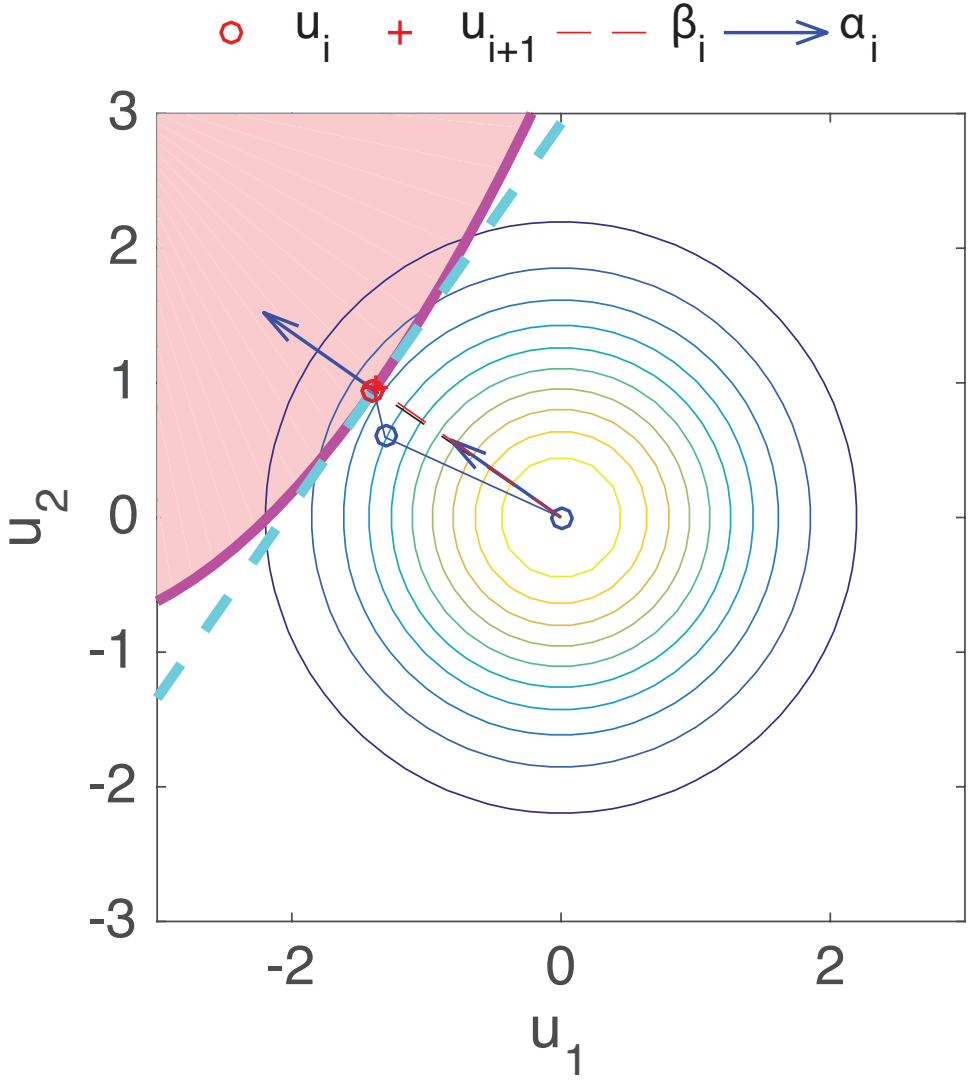
Exemple : Algorithme *iHL-RF*

$$\boxed{\hat{\boldsymbol{\alpha}}_i = -\frac{\nabla G(\mathbf{u}_i)}{\|\nabla G(\mathbf{u}_i)\|}}$$

$$\boxed{\beta_i \cong \frac{G(\mathbf{u}_i)}{\|\nabla G(\mathbf{u}_i)\|} + \hat{\boldsymbol{\alpha}}_i \mathbf{u}_i}$$

$$\boxed{\mathbf{u}_{i+1} = \beta_i \hat{\boldsymbol{\alpha}}_i^{\mathsf{T}}}$$

$i = 3$



Algorithme *iHL-RF* – facteur d'échelle

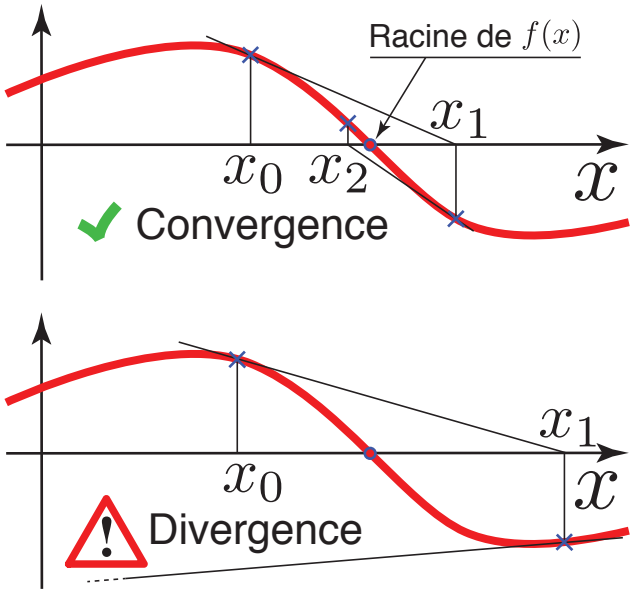
Les méthodes de type *Newton-Rapson* peuvent diverger si le pas $\|\mathbf{u}_{i+1} - \mathbf{u}_i\|$ est trop grand.

Solution:

$$\boxed{\mathbf{u}_{i+1} = \mathbf{u}_i + \lambda_i \mathbf{d}_i}$$

où le *facteur d'échelle* $\lambda_i \in (0, 1]$, et le *vecteur direction* est

$$\boxed{\mathbf{d}_i = \underbrace{\beta_i \hat{\boldsymbol{\alpha}}_i^{\mathsf{T}}}_{\mathbf{u}_{i+1}} - \mathbf{u}_i}$$



Algorithme *iHL-RF* – Vérification du facteur d'échelle

Pour $\mathbf{u}_{i+1} = \mathbf{u}_i + \lambda_i \mathbf{d}_i$, le choix d'une valeur de λ_i est adéquat si

$$\boxed{m(\mathbf{u}_i + \lambda_i \mathbf{d}_i) < m(\mathbf{u}_i)}$$

où

$$\boxed{m(\mathbf{u}) = \frac{1}{2}\|\mathbf{u}\|^2 + c|G(\mathbf{u})|}$$

et où c est une constante satisfaisant la contrainte

$$\boxed{c > \frac{\|\mathbf{u}_i\|}{\|\nabla G(\mathbf{u}_i)\|}}$$

La valeur de λ_i est optimale si

$$\boxed{\lambda_i = \arg\min_{\lambda}\{m(\mathbf{u}_i + \lambda \mathbf{d}_i)\}}$$

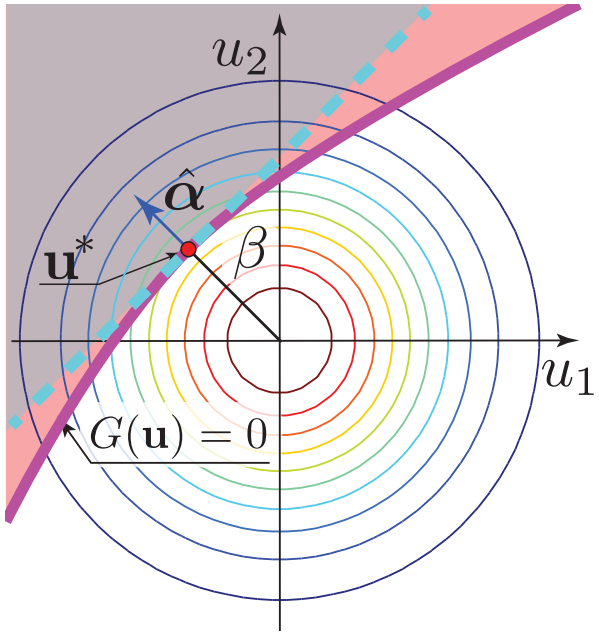
Algorithme *iHL-RF* – Critères de convergence

1 - $\mathbf{u}_i$ doit être sur la surface d'état limite

$$\boxed{\left|\frac{G(\mathbf{u}_i)}{G(\mathbf{0})}\right| < \epsilon_1}$$

2 - $\mathbf{u}_i$ doit être une projection de $\hat{\boldsymbol{\alpha}}$ à partir de l'origine

$$\boxed{\|\mathbf{u}_i - \overbrace{\underbrace{\hat{\boldsymbol{\alpha}}_i \mathbf{u}_i}_{\beta_i} \hat{\boldsymbol{\alpha}}_i^{\mathsf{T}}}^{\mathbf{u}_{i+1}}\| < \epsilon_2}$$

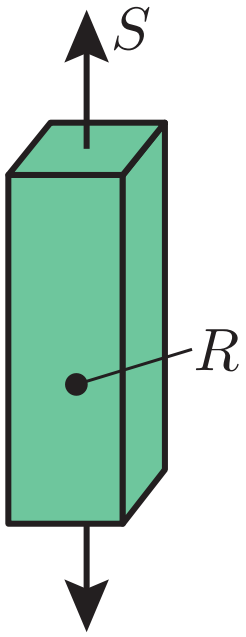


En pratique:
$\epsilon_1, \epsilon_2 \approx 10^{-3}$

Exemple #5.2

Soit $X_1 = D$ et $X_2 = S$ deux variables aléatoires où:

$$D \left\{ \begin{array}{lcl} \mu_D & = & 10\,\text{mm} \\ \sigma_D & = & 2\,\text{mm} \end{array} \right. , \; S \left\{ \begin{array}{lcl} \mu_S & = & 15\,\text{kN} \\ \sigma_S & = & 5\,\text{kN} \end{array} \right. , f_y = 300\,MPa, \; D \perp\!\!\!\perp S$$



$$\boxed{R(\mathbf{X}) = D^2 f_y, \; g(\mathbf{X}) = R(\mathbf{X}) - S}$$

$$\mathbf{X} = \left\{ \begin{array}{c} D \\ S \end{array} \right\}, \; \mathbf{M_X} = \left\{ \begin{array}{c} 10 \\ 15 \end{array} \right\}, \; \mathbf{D_X} = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}$$

$$\mathbf{R_{XX}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \; \mathbf{L} = \mathbf{L}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\nabla g(\mathbf{x}) = \left[\frac{\partial g(\mathbf{x})}{\partial x_1}, \frac{\partial g(\mathbf{x})}{\partial x_2} \right] = [2x_1 f_y, -1]$$

Exemple #5.2 (cont.)

Jabobien:

$$\mathbf{J}_{\mathbf{u},\mathbf{x}} = \mathbf{L}^{-1}\mathbf{D}_{\mathbf{X}}^{-1} = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.2 \end{bmatrix}, \quad \mathbf{J}_{\mathbf{x},\mathbf{u}} = \mathbf{J}_{\mathbf{u},\mathbf{x}}^{-1} = \mathbf{D}_{\mathbf{X}}\mathbf{L} = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}$$

Transformation $x \to u$:

$$\mathbf{u} = \underbrace{\mathbf{L}^{-1}\mathbf{D}_{\mathbf{X}}^{-1}}_{\mathbf{J}_{\mathbf{u},\mathbf{x}}}(\mathbf{x} - \mathbf{M}_{\mathbf{X}}) = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.2 \end{bmatrix}\left(\mathbf{x} - \begin{Bmatrix} 10 \\ 15 \end{Bmatrix}\right)$$

Transformation $u \to x$:

$$\mathbf{x} = \mathbf{M}_{\mathbf{X}} + \underbrace{\mathbf{D}_{\mathbf{X}}\mathbf{L}}_{\mathbf{J}_{\mathbf{x},\mathbf{u}}}\mathbf{u} = \begin{Bmatrix} 10 \\ 15 \end{Bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}\mathbf{u}$$

Exemple #5.2 (cont.)

Étapes à suivre:

1. Initialization: $\mathbf{x}_0 = \mathbf{M}_\mathbf{X} = [10, 15]^\mathsf{T} \to \mathbf{u}_0 = \mathbf{0} = [0, 0]^\mathsf{T}$

2. Évaluer la fonction d'état limite et son gradient
   $G(\mathbf{u}_i) = g(\mathbf{x}_i) = 10^2 f_y - 15 = 15$
   $\nabla G(\mathbf{u}_i) = \nabla g(\mathbf{x}_i)\mathbf{J}_{\mathbf{x},\mathbf{u}} = [12, -5]$

3. Calculer le vecteur $\hat{\boldsymbol{\alpha}}_i$
   $$\hat{\boldsymbol{\alpha}}_i = -\frac{\nabla G(\mathbf{u}_i)}{\|\nabla G(\mathbf{u}_i)\|} = -\frac{[12, -5]}{\sqrt{[12, -5][12, -5]^\mathsf{T}}} = [-0.92, 0.38]$$

4. Calculer le vecteur direction $\mathbf{d}_i$
   $$\mathbf{d}_i = \underbrace{\overbrace{\left[\frac{G(\mathbf{u}_i)}{\|\nabla G(\mathbf{u}_i)\|} + \hat{\boldsymbol{\alpha}}_i \mathbf{u}_i\right]}^{\beta_i} \hat{\boldsymbol{\alpha}}_i^\mathsf{T}}_{\mathbf{u}_{i+1}} - \mathbf{u}_i = [-1.07, 0.44]^\mathsf{T}$$

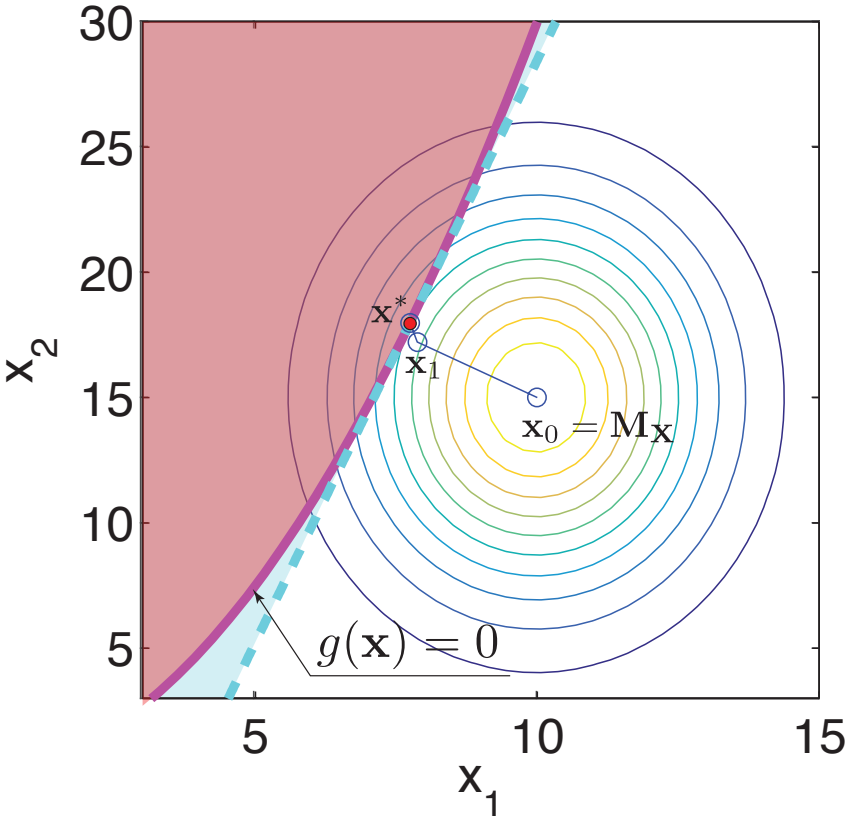
Exemple #5.2 (cont.)

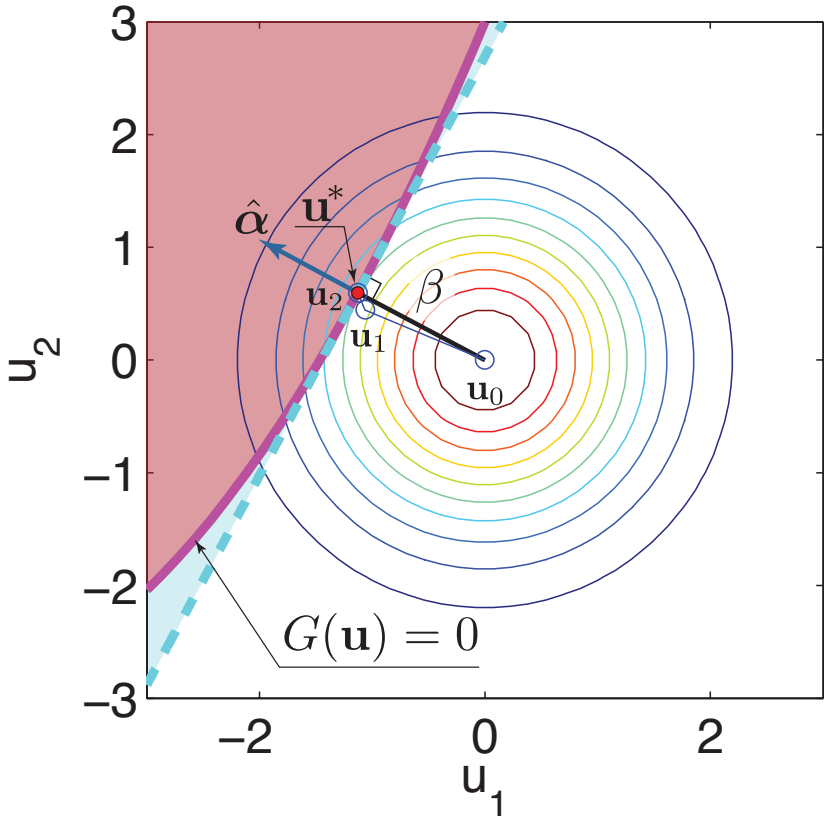
5. Calculer le point suivant avec $\lambda_0 = 1$
   $\mathbf{u}_{i+1} = \mathbf{u}_i + \lambda_i \mathbf{d}_i = [-1.07, 0.44]^\intercal$

6. Vérifier que la valeur λ_i utilisée est adéquate
   $\dfrac{\|\mathbf{u}_0\|}{\|\nabla G(\mathbf{u}_0)\|} = 0 \to c_0 := 10 > 0$
   $m(\mathbf{u}_0) = 1/2\|\mathbf{u}_0\|^2 + c_0|G(\mathbf{u}_o)| = 150$
   $m(\mathbf{u}_1) = 1/2\|\mathbf{u}_1\|^2 + c_0|G(\mathbf{u}_1)| = 14.3$
   Poursuivre si $m(\mathbf{u}_1) < m(\mathbf{u}_0)$, sinon, recommencer avec $\lambda_0 = \lambda_0/2$

7. Recommencer à l'étape 2 avec
   $\mathbf{x}_{i+1} = \mathbf{M_X} + \mathbf{D_X L u}_{i+1} = [7.87, 17.22]^\intercal$

Exemple #5.2 (cont.) [.m]





$$\beta = 1.28,\ \hat{\boldsymbol{\alpha}} = [-0.88, 0.47],\ \mathbf{x}^* = \begin{Bmatrix} 7.74 \\ 18.0 \end{Bmatrix},\ \mathbf{u}^* = \begin{Bmatrix} -1.13 \\ 0.60 \end{Bmatrix}$$

$$p_f = \mathbf{0.08},\ p_{f,\text{FOSM}} \approx 0.10,\ g_1 : p_{f,\text{MCFOSM}} \approx 0.12$$

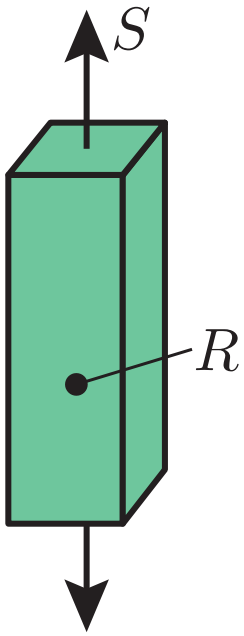
Exemple #5.2 (cont.) [.m]

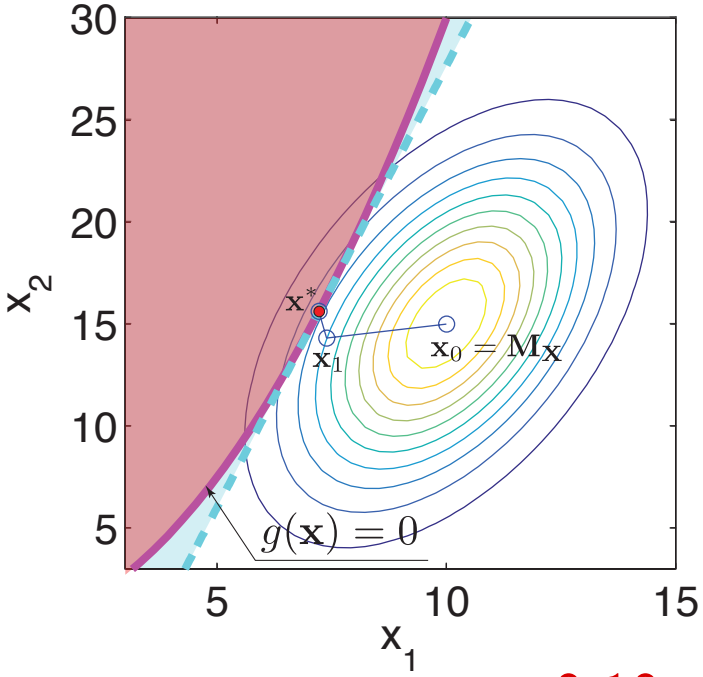
$$\lambda < 1$$

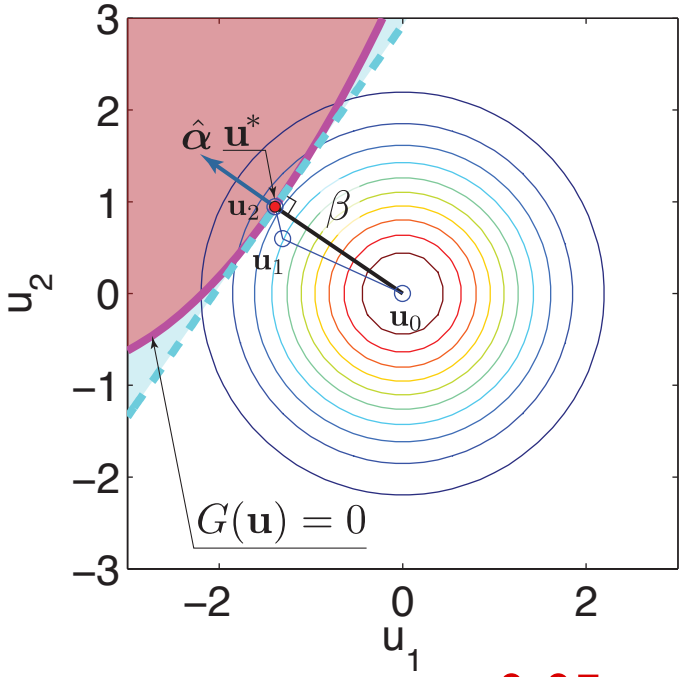
Exemple #5.2**b** [.m]

Soit $X_1 = D$ et $X_2 = S$ deux variables aléatoires où:

$$D \begin{cases} \mu_D &= 10\,\text{mm} \\ \sigma_D &= 2\,\text{mm} \end{cases}, \quad S \begin{cases} \mu_S &= 15\,\text{kN} \\ \sigma_S &= 5\,\text{kN} \end{cases}, \quad f_y = 0.3\,kN/m^2, \quad \rho_{DS} = 0.5$$



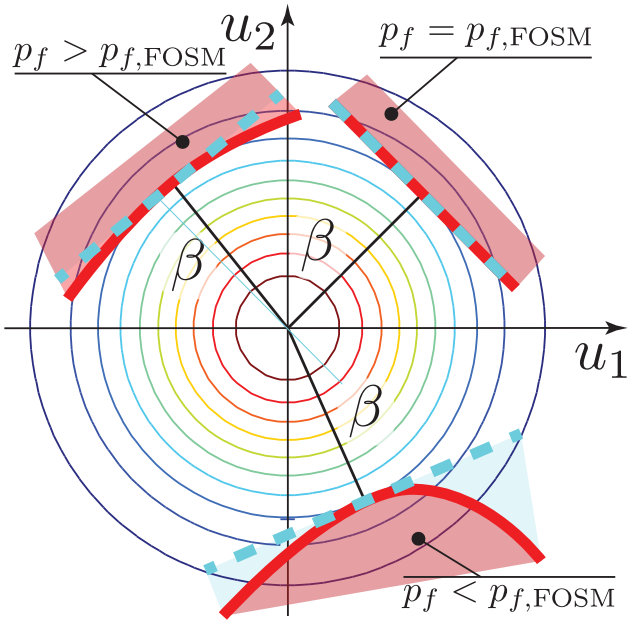




$\rho_{DS=0}$: $p_{f,\text{FOSM}} \approx 0.10$, $\rho_{DS=0.5}$: $p_{f,\text{FOSM}} \approx 0.05$

Résumé & limitations

- Les méthodes **MCFOSM** & **FOSM** ne prennent pas en compte la densité de probabilité conjointe de $f_{\mathbf{X}}(\mathbf{x})$. (applicable lorsque seulement $\mathbf{M_X}$ et $\mathbf{\Sigma_{XX}}$ sont connus)
- Les méthodes **MCFOSM** & **FOSM** approximent la fonction d'état limite $g(\mathbf{X})$ par un plan. Cette approximation peut être erronée si $g(\mathbf{X})$ est fortement non-linéaire.
- **MCFOSM** dépend de la formulation de la fonction d'état limite $g(\mathbf{X})$ et conduit à une estimation sub optimale de p_f.



Pour les 3 cas ci-haut:

- p_f différents
- $p_{f,\mathrm{FOSM}}$ égaux

Théorie probabilité

- 0 Révision algèbre & probabilités
- 1 Lois de probabilités
- 3 PDF multivariés

Fiabilité des structures & systèmes

- Introduction
- 2 Formulation fiabilité composantes $g_1(\mathbf{X})$
- 9 Formulation fiabilité systèmes

Estimation – p_f échantilonnage

- 4 Monte Carlo
- 12 Échantillonnage par importance

Estimation – p_f analytique

- 5 FOSM
- 6 FORM
- 8 SORM

Utilisation des résultats

- 7 Analyse de sensibilité
- 10 Incertitudes aléatoires & épistémiques
- 11 Données empiriques
- 13 Prise de décision
- 14 Métamodèles & modèles empiriques