

# Module #9

## Contenants sous pression et contraintes dues à la combinaison de chargements

(CIV1150 - Résistance des matériaux)

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Polytechnique Montréal

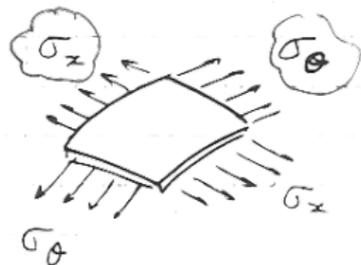
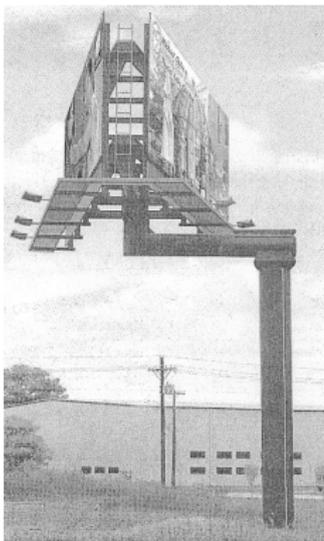


Sections 9.1-9.4 – R. Craig (2011)  
*Mechanics of Materials*, 3rd Edition  
John Wiley & Sons.

P. Léger (2006)  
*Notes de cours: Chapitre 9, §9.1–9.4*  
Polytechnique Montréal.

# Mise en contexte – chargement combiné

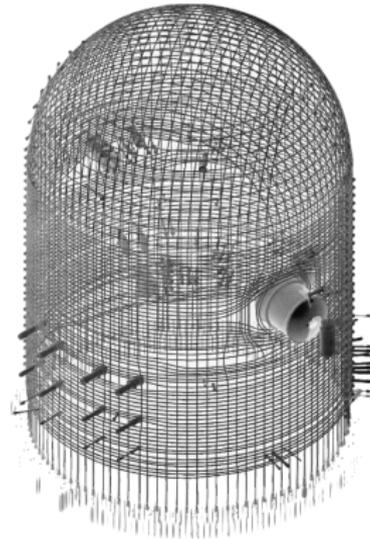
Contrainte maximale?



Génie civil?

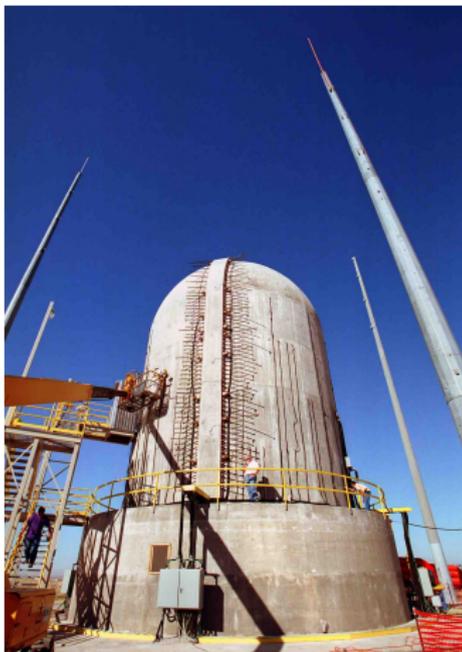
[Hibbler, 2005]

# Cuves sous pression – Réacteurs nucléaires



[France métallurgie, Kolombus]

# Cuves sous pression – Essai destructifs

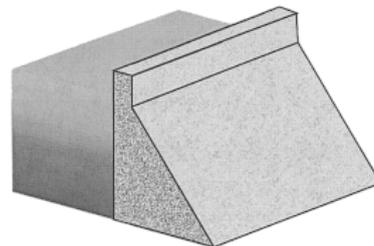


# Structures gonflables

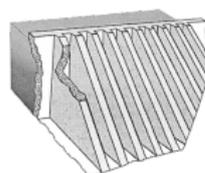


[Wikipedia, Pedretti (2004,2005), Google images]

# Barrages



(a) GRAVITY DAMS (3953)  
( $0 < H < 285\text{m}$ )



(b) BUTTRESS DAMS (337)  
( $0 < H < 130\text{m}$ )



(c) ARCH DAMS (1527)  
( $0 < H < 272\text{m}$ )



(d) MULTIPLE ARCH DAMS (126)  
( $0 < H < 214\text{m}$ )

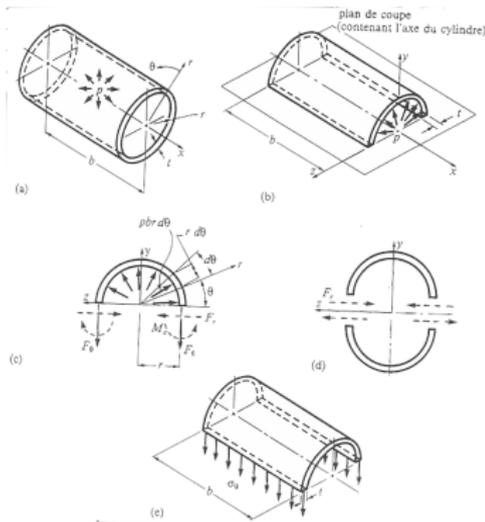
[Google images]

## Cylindres ouverts sous pression – contraintes

Force radiale:

– contraintes <sup>surface</sup>

$$F_r = p \overbrace{b(rd\theta)}$$



Composante verticale:

$$F_y(\theta) = F_r \cdot \sin \theta$$

Force circonférentielle:

$$\begin{aligned} F_\theta &= \frac{1}{2} \int_0^\pi F_y(\theta) d\theta \\ &= \frac{1}{2} [-bpr \cos \theta]_0^\pi \\ &= -\frac{1}{2} bpr [\cos \pi - \cos 0] \\ &= \boxed{bpr} \end{aligned}$$

Contrainte circonférentielle:

$$\sigma_\theta = \frac{F_\theta}{bt} = \frac{pr}{t}$$

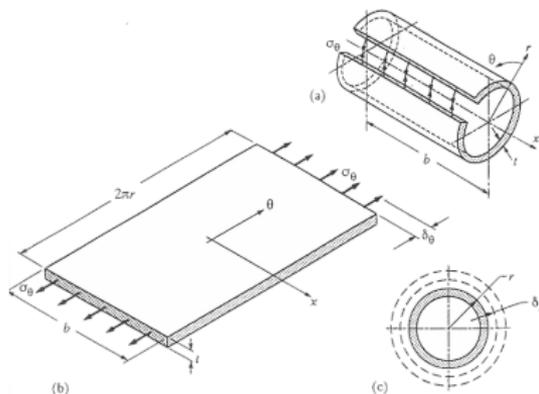
# Cylindres ouverts sous pression – déformations

Allongement (rappel):

$$\delta = \frac{FL}{EA}$$

Allongement circonférentiel:

$$\begin{aligned} \delta_{\theta} &= \frac{\sigma_{\theta} AL}{EA} = \frac{\sigma_{\theta} \cancel{A} L}{E \cancel{A}} \\ &= \frac{pr}{t} \cdot \frac{2\pi r}{E} = \frac{2\pi pr^2}{Et} \end{aligned}$$



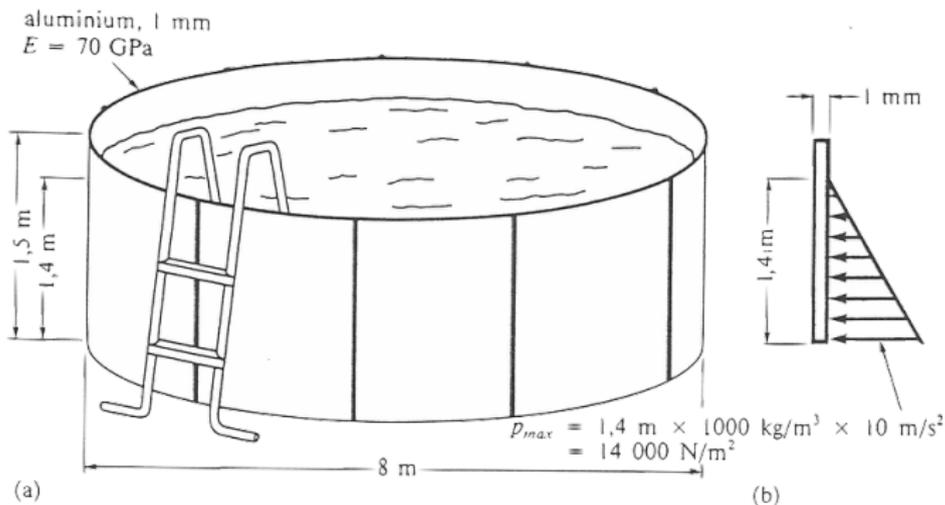
$$\sigma_{\theta} = \frac{pr}{t}$$

$$L = 2\pi r$$

Changement de rayon:

$$\delta_r = \frac{\delta_{\theta}}{2\pi} = \frac{pr^2}{Et}$$

# Cylindres ouverts sous pression – exemple



$$\sigma_{\theta} = \frac{pr}{t}$$

$$\delta_r = \frac{pr^2}{Et}$$

**Calculer  $\sigma_{\theta, \max}$  et  $\delta_{\theta \max}$**

# Cylindres fermés sous pression – contraintes

Contrainte circonférentielle:

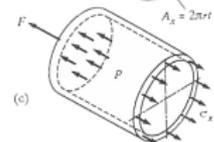
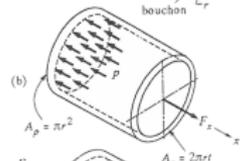
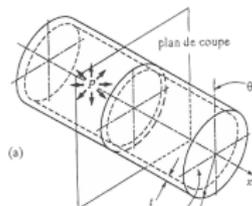
$$\sigma_{\theta} = \frac{pr}{t}$$

Force sur le bouchon:

$$F_x = \pi r^2 p$$

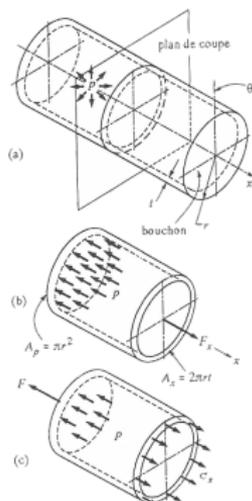
Contrainte axiale:

$$\sigma_x = \frac{F_x}{A_x} = \frac{\pi r^2 p}{2\pi r t} = \frac{pr}{2t}$$



**État de contraintes bi-axiales,  $\sigma_{\theta} = 2\sigma_x$**

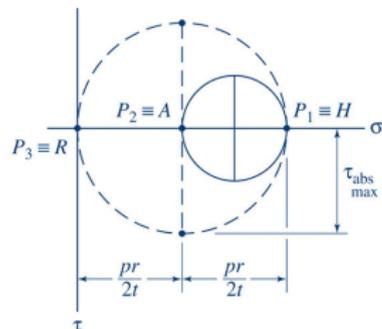
## Cylindres fermés sous pression – Mohr



**État de contraintes bi-axiales,  $\sigma_\theta = 2\sigma_x$**

A = axial direction  
H = hoop direction  
R = radial direction

—  $p_1 p_2$  (in-plane)  
Mohr's circle  
- - -  $p_1 p_3$  (out-of-plane)  
Mohr's circle

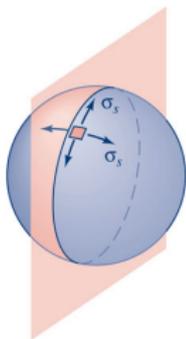


Contrainte de cisaillement maximale:

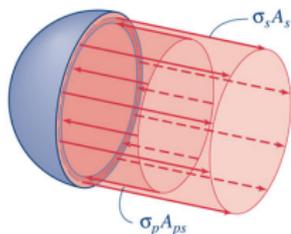
$$|\tau_{max}| = \frac{\sigma_1 - \sigma_3}{2} = \frac{1}{2} \left( \frac{2pr}{2t} \right) = \boxed{\frac{pr}{2t}} = \sigma_x$$

[Bazergui]

# Sphères sous pression – Contraintes



(a) A spherical pressure vessel with diametral cutting plane.



Force dans le liquide/gaz

$$F_p = p \cdot \pi r^2$$

Force dans la paroi

$$F_s = \sigma \cdot A = \sigma \cdot 2\pi r t$$

Équilibre des forces

$$F_s \equiv F_p$$

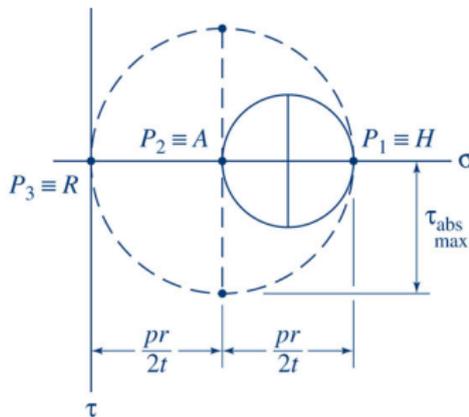
Contrainte

$$\sigma_s = \frac{pr}{2t} = \sigma_x$$

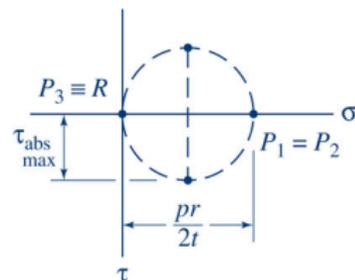
# Comparaison sphères v.s. cylindres

A = axial direction  
 H = hoop direction  
 R = radial direction

—  $p_1 p_2$  (in-plane)  
 Mohr's circle  
 - - -  $p_1 p_3$  (out-of-plane)  
 Mohr's circle



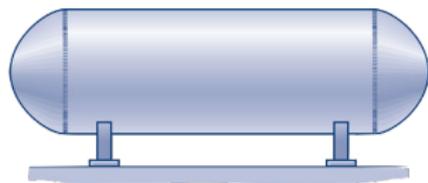
(a) Mohr's circles for a cylinder.



(b) Mohr's circles for a sphere.

$$|\tau_{\max, \text{sphere}}| = \frac{\sigma_1 - \sigma_3}{2} = \frac{pr}{4t} = \frac{1}{2} |\tau_{\max, \text{cylindre}}|$$

## Cylindre fermé sous pression – exemple 🧩



- ▶  $D_i = 1\,200\text{ mm}$
- ▶  $p = 2\text{ MPa}$
- ▶  $\sigma_{adm.} = 100\text{ MPa}$
- ▶  $\tau_{adm.} = 40\text{ MPa}$

**Calculer l'épaisseur minimale du réservoir**

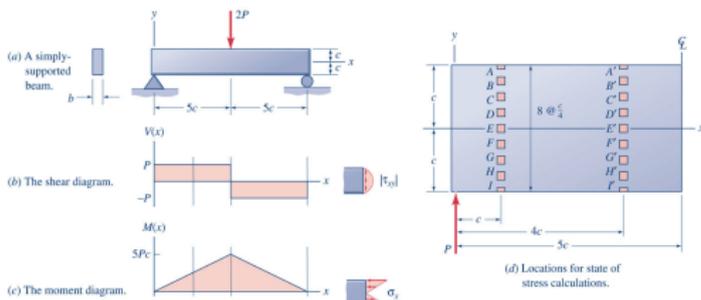
# Cylindre fermé sous pression – Défaillance

?

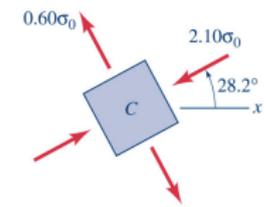
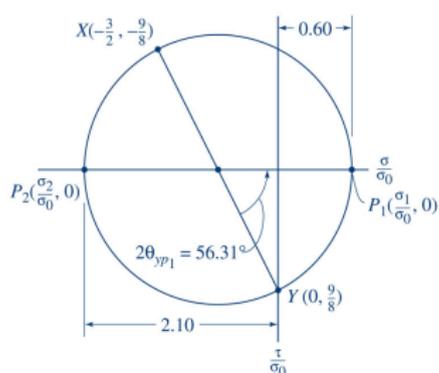
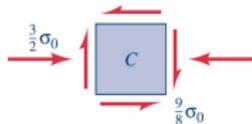


Distribution des contraintes dans les poutres

# Distribution des contraintes dans les poutres

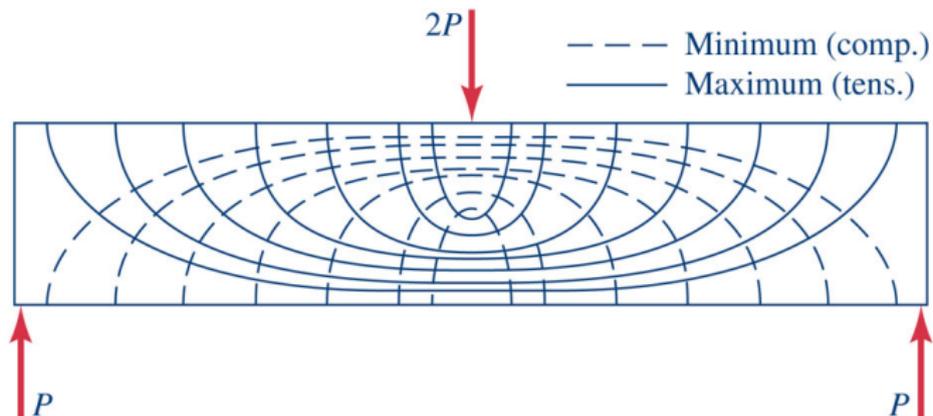


$$\sigma_0 \equiv \frac{P}{A}$$

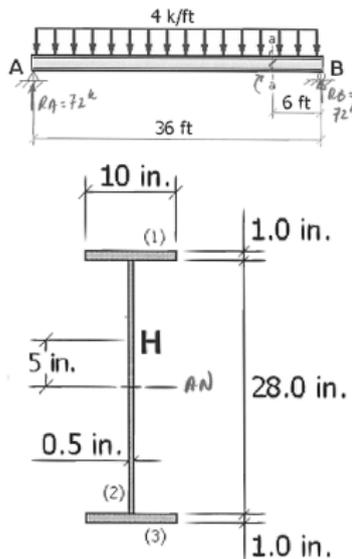


y/c	x = c		x = 4c	
	$\sigma_{xx}/\sigma_0$	$\sigma_{yy}/\sigma_0$	$\sigma_{xx}/\sigma_0$	$\sigma_{yy}/\sigma_0$
1	-3.00	0	-12.00	0
3/4	-2.43	0.18	-9.05	0.05
1/2	-2.10	0.60	-6.20	0.20
1/4	-1.83	1.08	-3.56	0.56
0	-1.50	1.50	-1.50	1.50
-1/4	-1.08	1.83	-0.56	3.56
-1/2	-0.60	2.10	-0.20	6.20
-3/4	-0.18	2.43	-0.05	9.05
-1	0	3.00	0	12.00

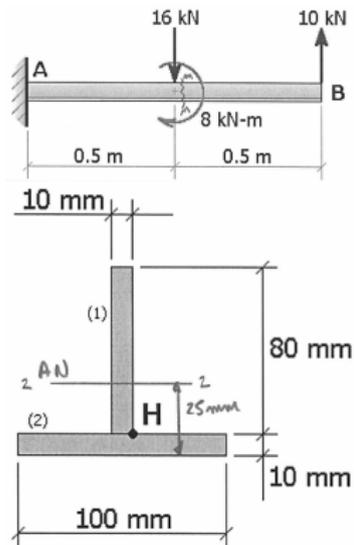
# Champ de contraintes dans les poutres



## Exemple MDS 9.2

**Question:**

- Calculer  $V$  et  $M$  à la section a-a
- Calculer  $\sigma_x$  et  $\tau_y$  au point H
- Calculer  $\sigma_1$  et  $|\tau_{\max}|$  au point H

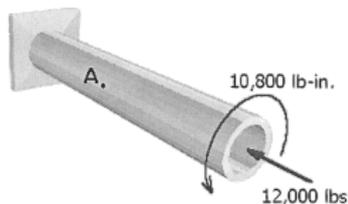
Exemple MDS 9.3 

$$I_z = 1\,335\,000 \text{ mm}^4$$

**Question:**

Pour la section M-M et pour le point H

- Calculer  $V$  et  $M$
- Calculer  $\sigma_x$  et  $\tau_{xy}$
- Calculer  $\sigma_1$  et  $|\tau_{\max}|$

Exemple MDS 9.4 

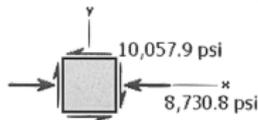
$$D_{ext} = 2 \text{ in}$$

$$D_{int} = 1.5 \text{ in}$$

$$A = \frac{\pi}{4} (D_{ext}^2 - D_{int}^2) = 1.37 \text{ in}^2$$

$$I_p = \frac{\pi}{32} (D_{ext}^4 - D_{int}^4) = 1.07 \text{ in}^4$$

**Question :**  
Calculer  $\sigma_1$  et  $|\tau_{\max}|$  au point H



$$\sigma_{axial} = F/A = \frac{-12000 \text{ lb}}{1.3744 \text{ in}^2} = -8730.8 \text{ psi COMP}$$

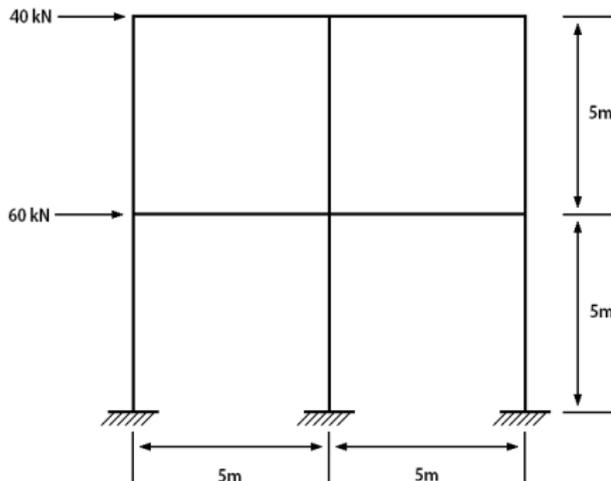
$$\gamma = \frac{T \cdot r}{I_p} = \frac{(10,800 \text{ lb-in})(1.0 \text{ in})}{1.0737 \text{ in}^4}$$

$\gamma = 10057.9 \text{ psi}$  agit vers le Bas sur la facette 4 au point A

# Formules pour le calcul des contraintes

Effort	Symbole	Formule	[Craig.]
Force normale	$F$	$\sigma = \frac{F}{A}$	§2.2, 3.2
Moment de torsion	$T$	$\tau = \frac{T\rho}{I_p}$	§4.3
Moment de flexion	$M$	$\sigma = -\frac{My}{I}$	§6.3
Force de cisaillement	$V$	$\tau = \frac{VQ}{It}$	§6.8

# Exemple d'application: cadres rigides

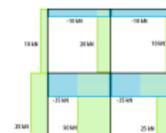


**Dans le cadre du cours on s'intéresse à la combinaison  $N-M$**

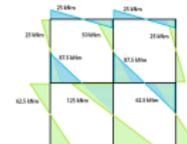
## Efforts axiaux – $N$



## Efforts tranchants – $V$



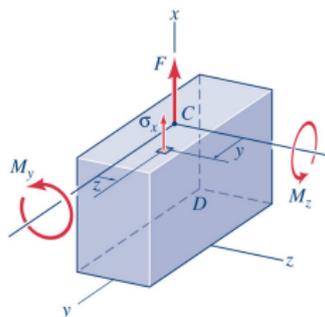
## Moments fléchissants – $M$



# Hypothèses pour combinaison des chargements

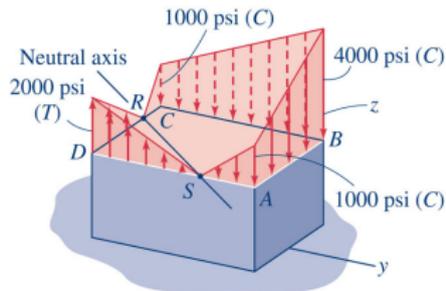
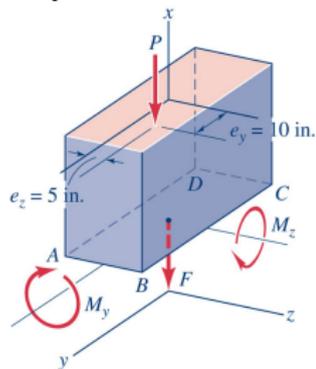
1. Matériau linéaire et élastique
2. Petites déformations et petits déplacements
3. Section et structure stables  
( $\Delta$ flambement,  $\Delta$ voilement,  $\Delta$ déversement,  $\Delta p - \Delta$ )

# Flexion et chargement axial



$$\sigma_x = \frac{F}{A} + \frac{M_y z}{I_y} - \frac{M_z y}{I_z}$$

## Exemple: flexion et chargement axial

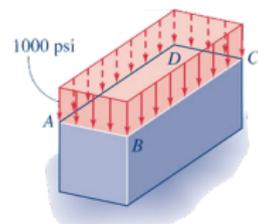
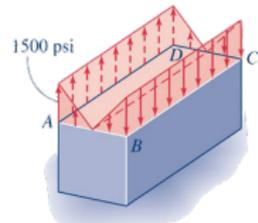
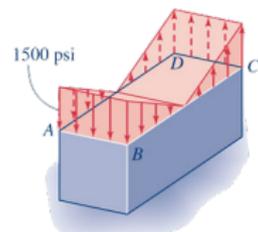


$$F = -P$$

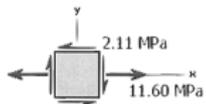
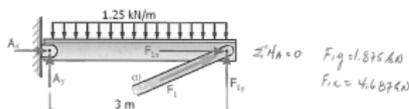
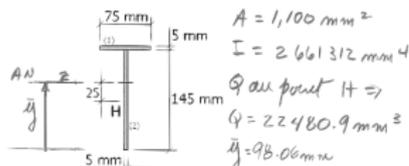
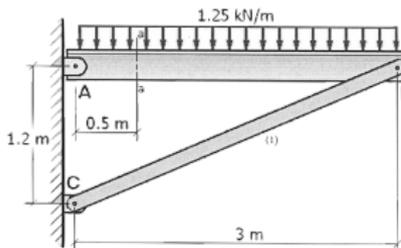
$$M_y = -Pe_z$$

$$M_z = Pe_y$$

$$\sigma_x = \frac{F}{A} + \frac{M_y z}{I_y} - \frac{M_z y}{I_z}$$

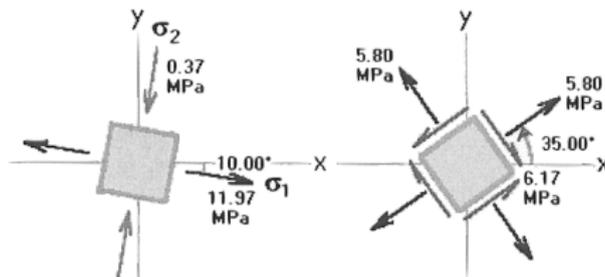
(a) Stress distribution due to  $F$ .(b) Stress distribution due to  $M_y$ .(c) Stress distribution due to  $M_z$ .

# Exemple MDS 9.5



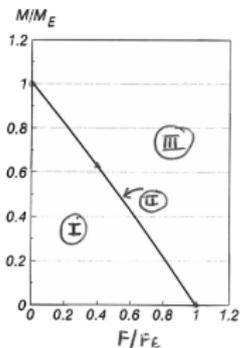
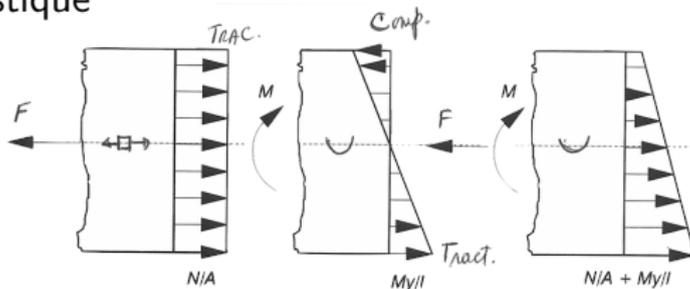
## Question:

Calculer  $\sigma_x$  et  $\tau_{xy}$  au point H situé au droit de la coupe a-a



# Courbes d'interaction $F-M$

## Domaine élastique



- Ⓘ COMPORTEMENT ELASTIQUE
- Ⓜ COMBINAISON DE M et F pour le début de la plasticification du fibres
- Ⓝ comportement plastique ou rupture.

$$\frac{F}{A\sigma_y} + \frac{Mc}{I_z\sigma_y} = 1$$

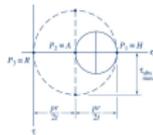
$$M_E = \frac{I_z\sigma_y}{c}, \quad F_E = A\sigma_y$$

$$\frac{M}{M_E} = 1 - \frac{F}{F_E}$$

[Filiatreau, 1996]

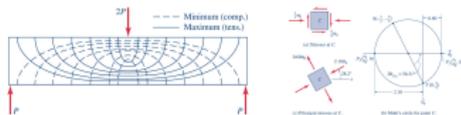
# Résumé – Module #9

## Cylindres fermés sous pression (contraintes)

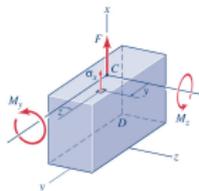


$$\begin{array}{l}
 \sigma_{\theta} = 2\sigma_x \\
 |\tau_{max}| = \sigma_x
 \end{array}
 \left\{
 \begin{array}{l}
 \text{circonférentielle:} \\
 \text{axiale:} \\
 \text{de cisaillement maximale:}
 \end{array}
 \right.
 \begin{array}{l}
 \sigma_{\theta} = \frac{pr}{t} \\
 \sigma_x = \frac{pr}{2t} \\
 |\tau_{max}| = \frac{pr}{2t}
 \end{array}$$

## Distribution des contraintes dans les poutres



## Combinaison de chargement en flexion et axial



$$\sigma_x = \frac{F}{A} + \frac{M_y z}{I_y} - \frac{M_z y}{I_z}$$

## Organisation de la matière

- |                    |                 |   |   |
|--------------------|-----------------|---|---|
| 1                  | <i>Statique</i> | { | <ul style="list-style-type: none"> <li>- Équilibre des forces et moments</li> <li>- Diagrammes de corps libres </li> <li>- 5 Diagramme des efforts, <math>N(x)</math>, <math>V(x)</math>, <math>M(x)</math></li> </ul>   |
| 2                  | <i>Matériau</i> | { | <ul style="list-style-type: none"> <li>- Contraintes &amp; déformations</li> <li>- Loi de Hooke, Poisson &amp; St-Venant </li> </ul>   |
| <i>Chargements</i> |                 | { | <ul style="list-style-type: none"> <li>- 3 Efforts axiaux </li> <li>- 4 Torsion </li> <li>- 6a Flexion </li> <li>- 6b Cisaillement </li> <li>- 7 Déflexion </li> <li>- 9 Pression &amp; chargements combinés </li> </ul> |
|                    |                 | { | <ul style="list-style-type: none"> <li>- 7 Déflexion </li> </ul>   |
|                    |                 | { | <ul style="list-style-type: none"> <li>- 8 Contraintes 2D-3D </li> </ul>   |
|                    |                 | { | <ul style="list-style-type: none"> <li>- 10 Lois constitutives &amp; critères de rupture </li> </ul>  |
|                    |                 | { |   |