Module #8b
Transformation des contraintes et des déformations 2D-3D : Cercle de Mohr

(CIV1150 - Résistance des matériaux)

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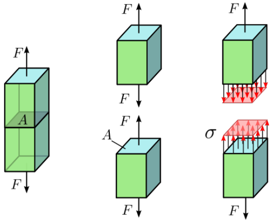
Sections 8.5-8.10 – R. Craig (2011)
Mechanics of Materials, 3rd Edition
John Wiley & Sons.

P. Léger (2006)
Notes de cours: Chapitre 8
Polytechnique Montréal.

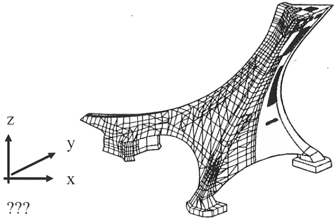
Mise en contexte (Rappel Module 8a)

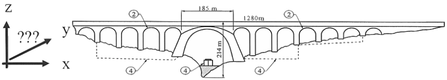
Contrainte dans la barre?



$$\sigma = \frac{F}{A}$$

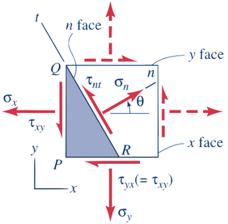
Contrainte dans la structure?





Requiert l'analyse des contraintes principales en 2D-3D

Rappel Module 8a



Formules rotation du système d'axe

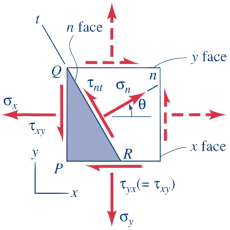
$$\sigma_n = \left(\frac{\sigma_x+\sigma_y}{2}\right) + \left(\frac{\sigma_x-\sigma_y}{2}\right)\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\tau_{nt} = -\left(\frac{\sigma_x-\sigma_y}{2}\right)\sin 2\theta + \tau_{xy}\cos 2\theta$$

Module 8b : cercle de Mohr
$\rightarrow$ méthode graphique pour calculer :

$$\underbrace{\sigma_n, \tau_{nt}}_{\text{contraintes normales et tangentielles}}, \underbrace{\sigma_1, \sigma_2, \theta_p}_{\text{contraintes principales}}, \underbrace{\tau_{s1}, \theta_s}_{\text{cisaillement maximal}}$$

Formules rotation du système d'axe



Formules rotation du système d'axe

$$\sigma_n - \sigma_{avg.} = \left(\frac{\sigma_x - \sigma_y}{2}\right)\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\tau_{nt} = -\left(\frac{\sigma_x - \sigma_y}{2}\right)\sin 2\theta + \tau_{xy}\cos 2\theta$$

$$R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$(\sigma_n - \sigma_{avg.})^2 + \tau_{nt}^2 = \left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2$$

$$= R^2$$

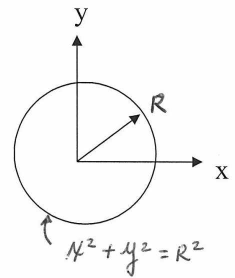
$$\underbrace{(\sigma_n - \sigma_{avg.})^2 + \tau_{nt}^2 = R^2}_{\text{Cercle!}}$$

Représentation graphique – Cercle de Mohr

$$\boxed{(\sigma_n - \sigma_{avg.})^2 + \tau_{nt}^2 = R^2}$$

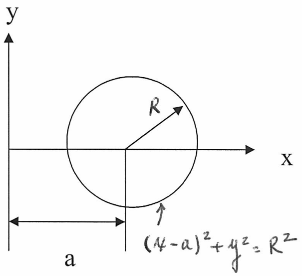
Rayon du cercle:

$$R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

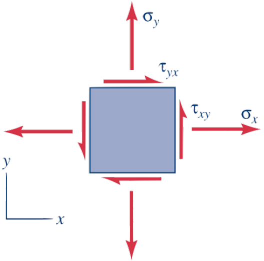


Centre du cercle:

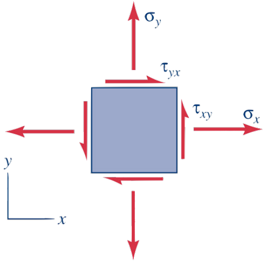
$$\sigma_{avg.} = \frac{\sigma_x + \sigma_y}{2}$$



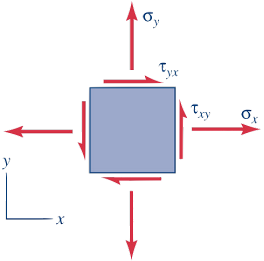
Cercle de Mohr

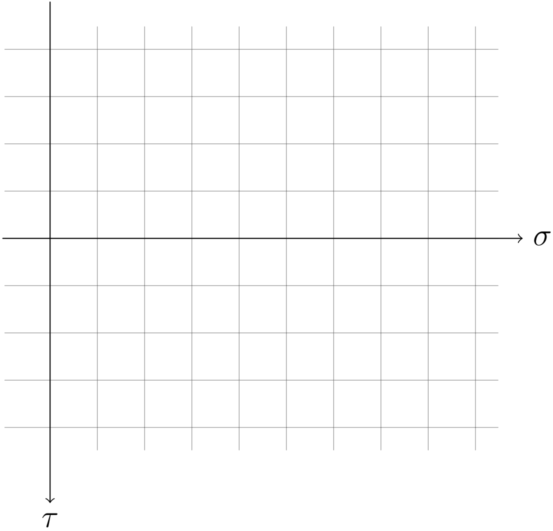


Cercle de Mohr

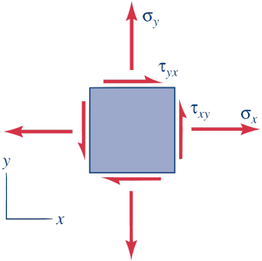


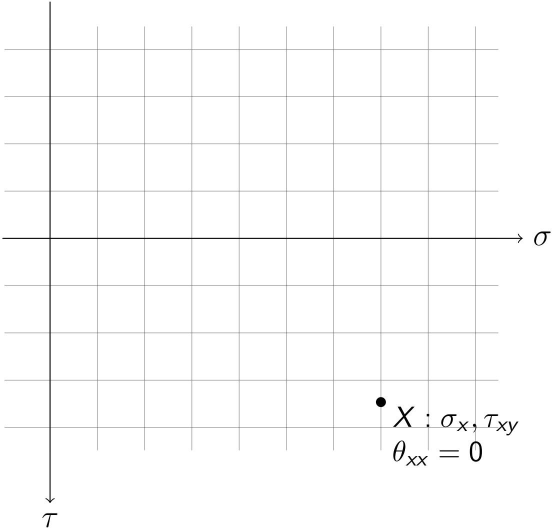
Cercle de Mohr



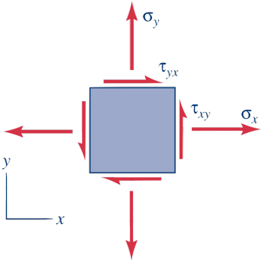


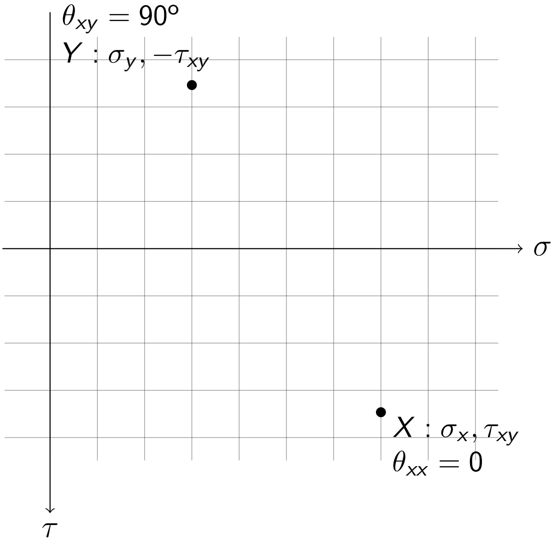
Cercle de Mohr



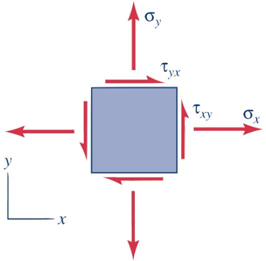


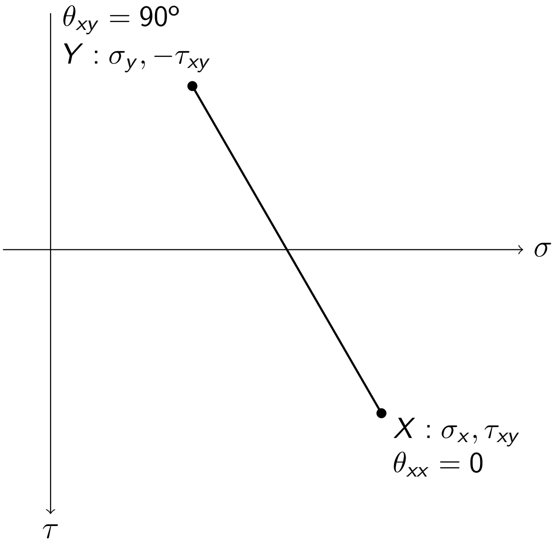
Cercle de Mohr



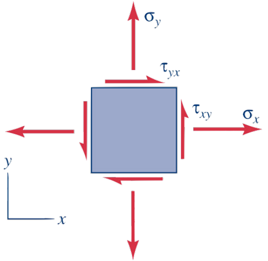


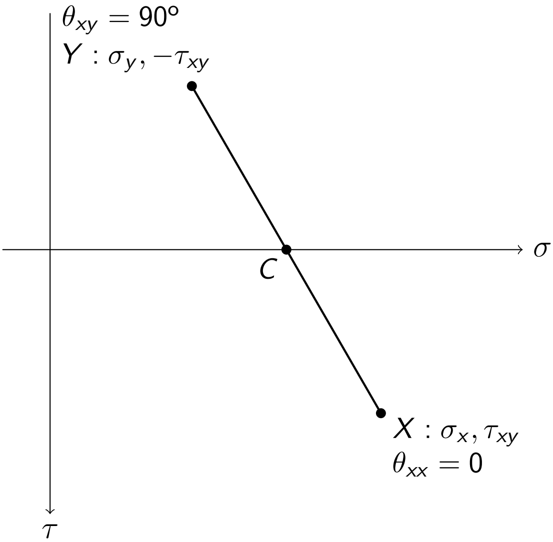
Cercle de Mohr



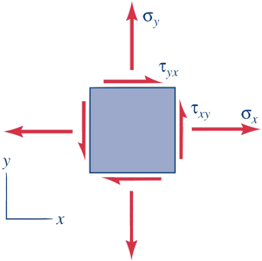


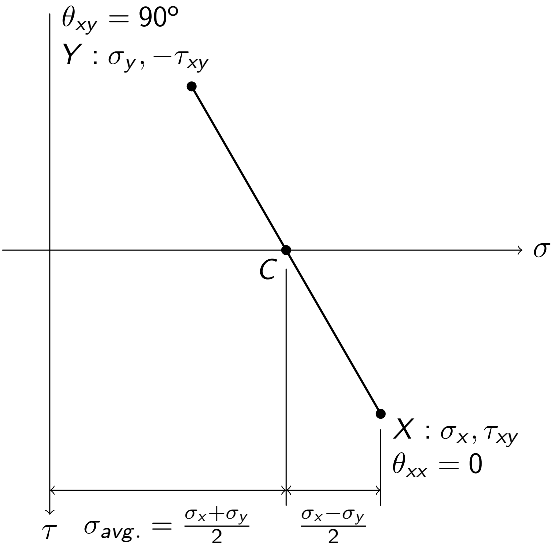
Cercle de Mohr



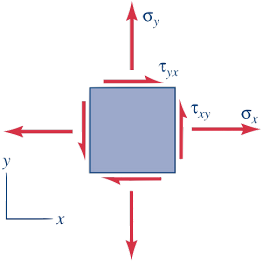


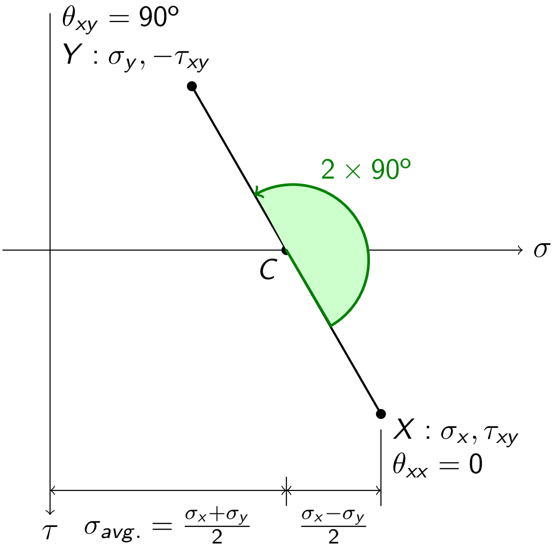
Cercle de Mohr



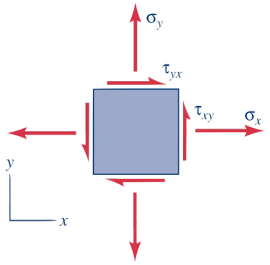


Cercle de Mohr



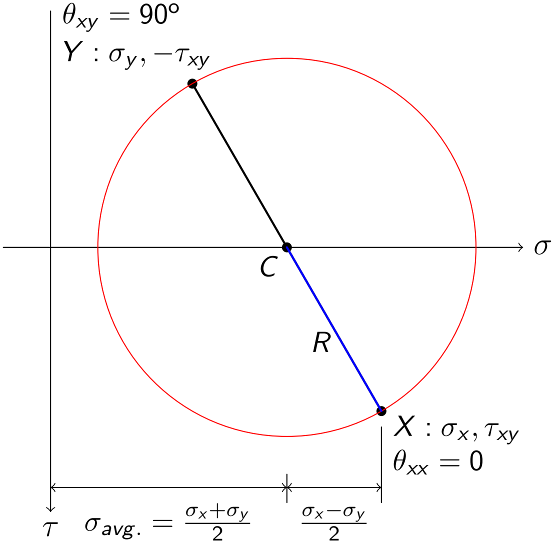


Cercle de Mohr

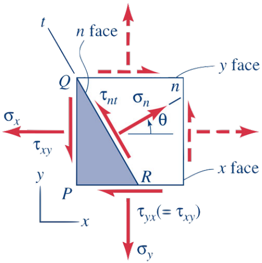


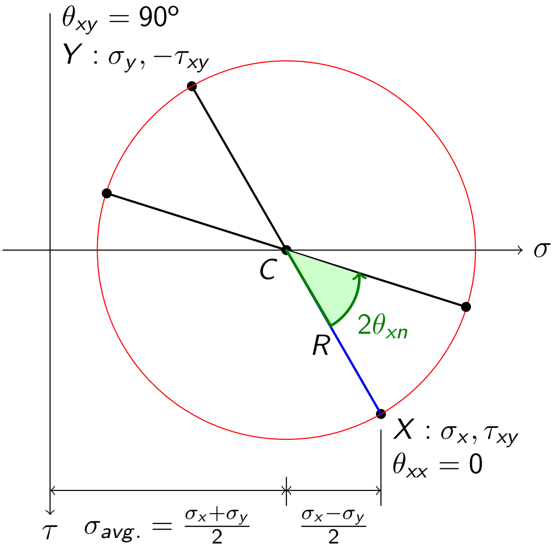
$$(\sigma_n - \sigma_{avg.})^2 + \tau_{nt}^2 = R^2$$

$$R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

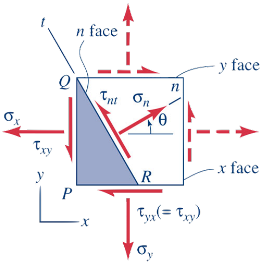


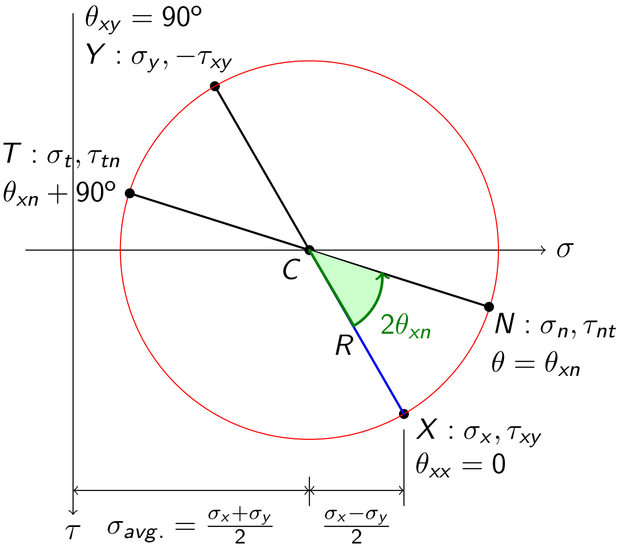
Cercle de Mohr



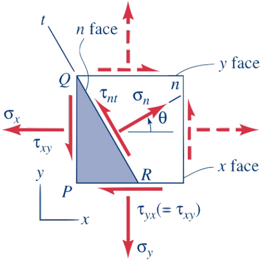


Cercle de Mohr

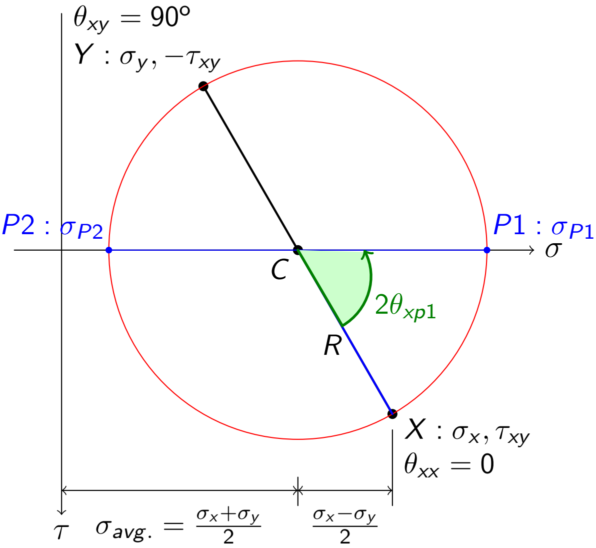




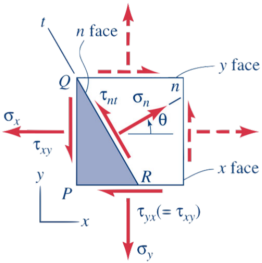
Cercle de Mohr

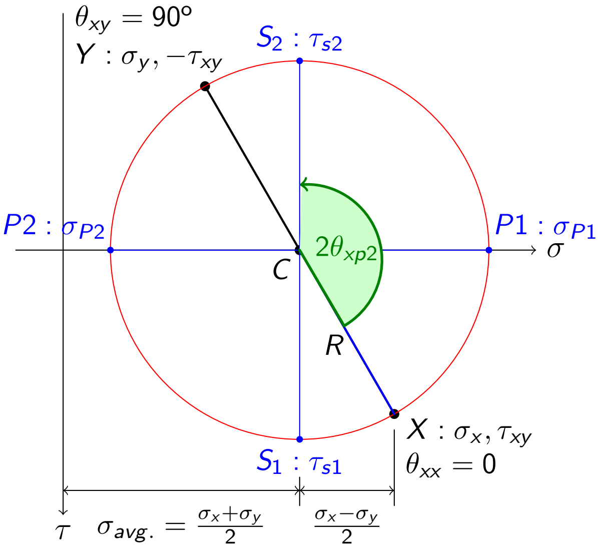


$$\tan 2\theta_{p1} = \frac{\tau_{xy}}{\frac{\sigma_x - \sigma_y}{2}}$$



Cercle de Mohr

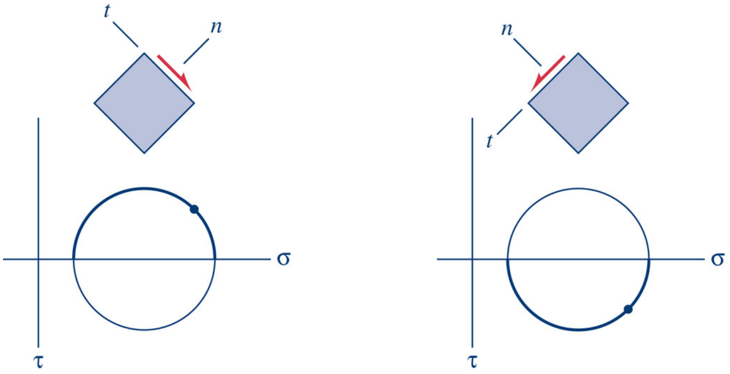




Étapes de construction du cercle de Mohr

1. Identifier les contraintes ($\pm$) agissant sur l'élément dans le repère $X - Y$  (⚠ τ + si rotation anti-horaire)
2. Tracer les axes (⚠ même échelle & τ + vers le bas)
3. Marquer les points $X : (\sigma_x, \tau_{xy})$ et $Y : (\sigma_y, \tau_{yx} = -\tau_{xy})$
4. Tracer le segment XY, et marquer le centre C
5. Tracer le cercle de centre C et passant par X et Y
6. $N(\sigma_n, \tau_{nt})$: Rotation de X vers N dans le même sens pour le cercle de Mohr et l'élément (⚠ angle θ doublé sur le cercle)
7. σ principales $P(\sigma_1, 0) : \sigma_1 = \sigma_{avg.} + R$, $\sigma_2 = \sigma_{avg.} - R$
8. τ maximal $S(\sigma_{avg.}, \tau_{s1}) : \sigma_{s1} = \sigma_{avg.}$, $\sigma_{s2} = \sigma_{avg.}$

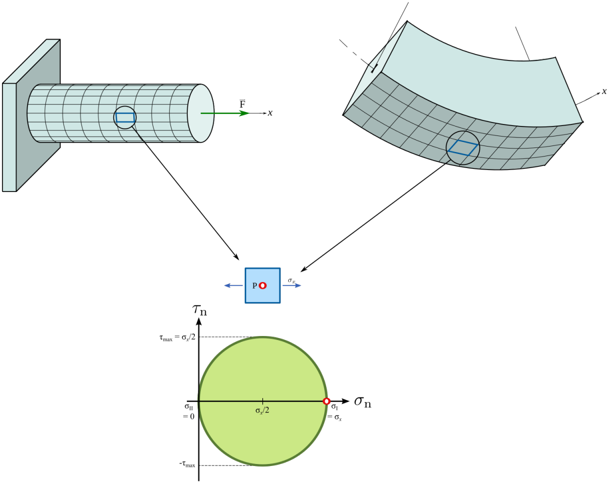
Signes & cisaillement



(*a*) Clockwise shear stress. (*b*) Counterclockwise shear stress.

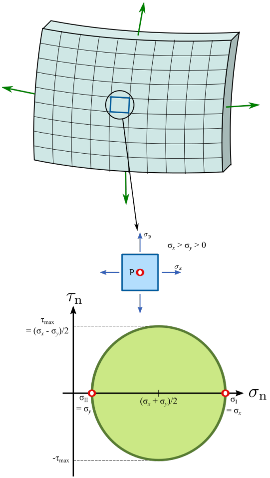
+ : rotation **anti-horaire**

Exemple – Contrainte uniaxiale (⚠ signe inversé)



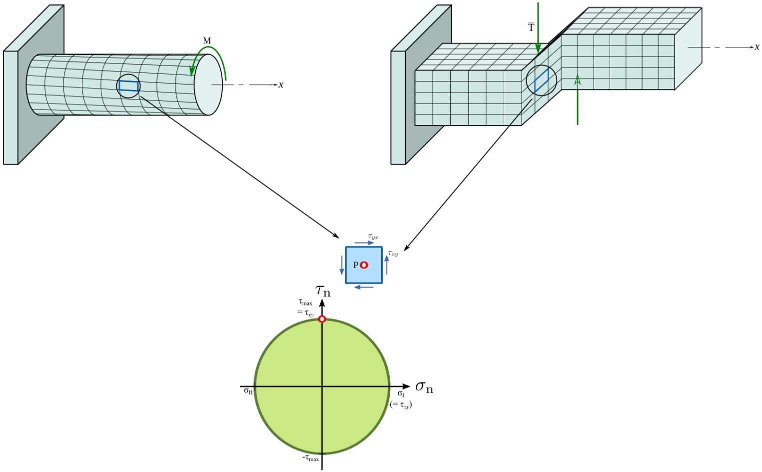
[Wikipedia]

Exemple – Contrainte biaxiale (⚠ signe inversé)



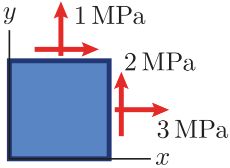
[Wikipedia]

Exemple – Cisaillement pur (⚠ signe inversé)



[Wikipedia]

Exemple 1



$$X(\sigma_x = +3\ MPa, \tau_{xy} = +2\ MPa)$$

$$Y(\sigma_y = +1\ MPa, \tau_{yx} = -2\ MPa)$$

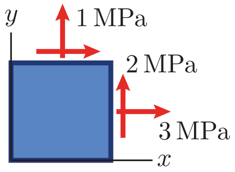
$$\sigma_{avg.} = \frac{\sigma_x+\sigma_y}{2} = \frac{3\ MPa+2\ MPa}{2} = 2\ MPa$$

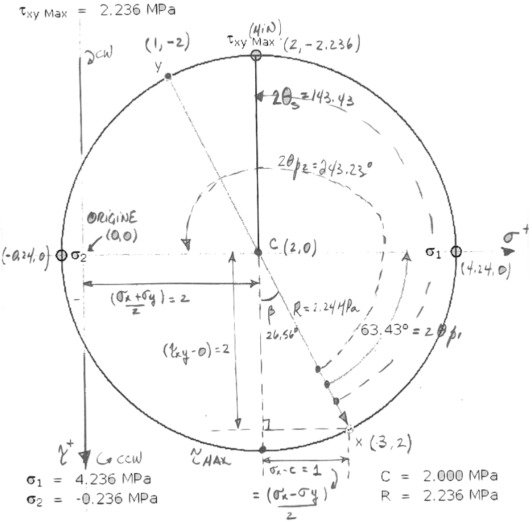
$$R = \sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2 + \tau_{xy}^2} = \sqrt{\left(\frac{3\ MPa-1\ MPa}{2}\right)^2 + 2\ MPa^2} = 2.24\ MPa$$

$$\sigma_{p1} = \sigma_{avg.} + R = 4.24\ MPa,\ \sigma_{p2} = \sigma_{avg.} - R = -0.24\ MPa$$

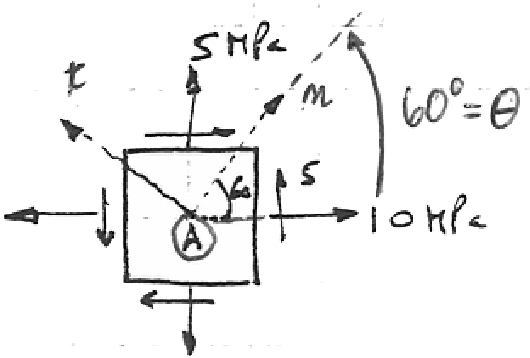
$$\tan 2\theta_{xp1} = \frac{\tau_{xy}}{\frac{\sigma_x-\sigma_y}{2}} \rightarrow \theta_{xp1} = \frac{1}{2}\arctan\left(\frac{2\ MPa}{\frac{3\ MPa-1\ MPa}{2}}\right) = 31.72^\circ$$

Exemple 1 (cont.)

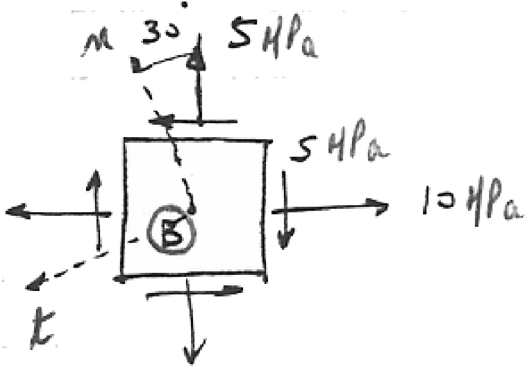




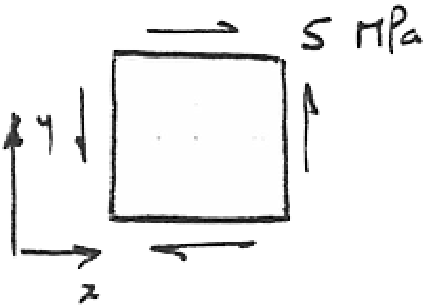
Exemple 2



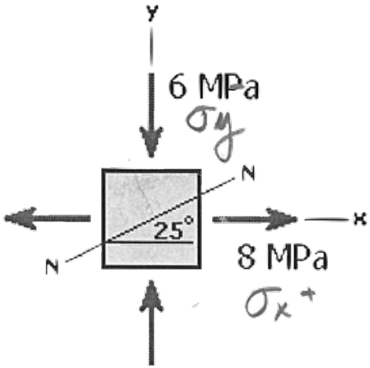
Exemple 3



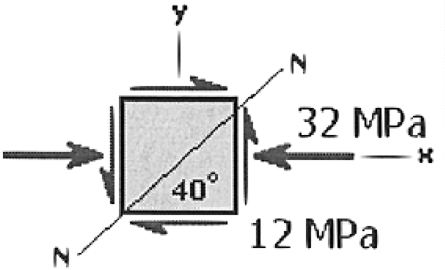
Exemple 4



Exemple 5

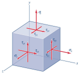


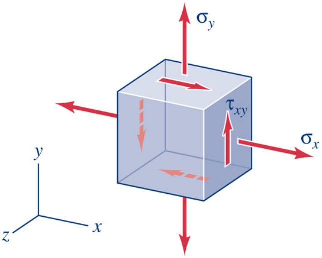
Exemple 6



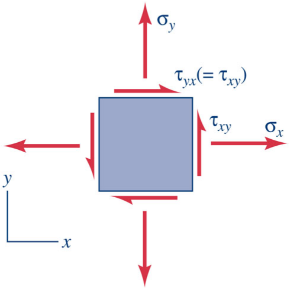
Contraintes planes (rappel)

Contraintes planes: $\sigma_z = \tau_{xz} = \tau_{yz} = 0$



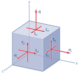


(a) Three-dimensional view.

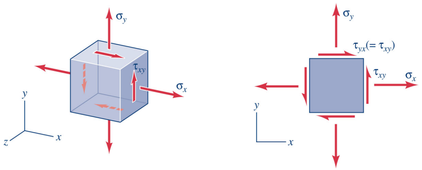


(b) Two-dimensional view.

Contraintes planes (rappel)



Contraintes planes: $\sigma_z = \tau_{xz} = \tau_{yz} = 0,\ \epsilon_z \neq 0$

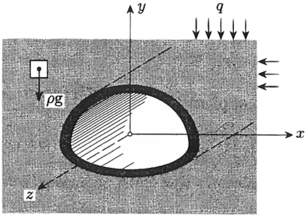


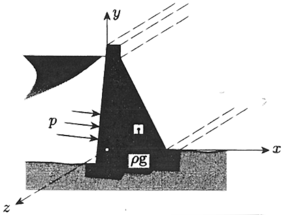
(*a*) Three-dimensional view. (*b*) Two-dimensional view.

$$\boxed{\begin{aligned}\epsilon_x &= \frac{1}{E}(\sigma_x - \nu\sigma_y)\\ \epsilon_y &= \frac{1}{E}(\sigma_y - \nu\sigma_x)\\ \epsilon_z &= -\frac{\nu}{E}(\sigma_x + \sigma_y)\end{aligned}}$$

Déformations planes

Déformations planes: $\epsilon_z = \gamma_{xz} = \gamma_{yz} = 0$

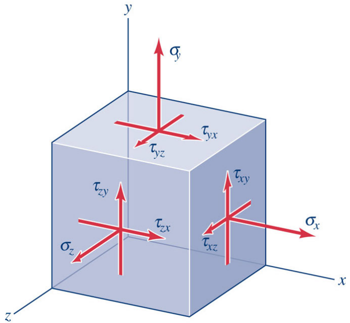




$$\boxed{\epsilon_z = 0 \rightarrow \sigma_z = \frac{E\nu}{(1+\nu)(1-\nu)}(\epsilon_x + \epsilon_y)}$$

[Frey, 1998]

Tenseur de contraintes 3D



Tenseur de contraintes

i: Plan normal, j: Direction

$$[\sigma_{ij}] = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix}$$

Afin de satisfaire les éq. d'équilibre:

$$\tau_{xy} = \tau_{yx},\ \tau_{xz} = \tau_{zx},\ \tau_{yz} = \tau_{zy}$$

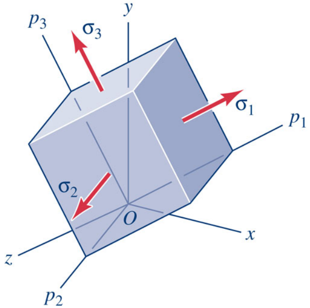
Contraintes principales

$$\sigma_1 \geq \sigma_2 \geq \sigma_3$$

$$\begin{aligned} \sigma_1 &\equiv \sigma_{\text{max}} \\ \sigma_2 &\equiv \sigma_{\text{int}} \\ \sigma_3 &\equiv \sigma_{\text{min}} \end{aligned}$$

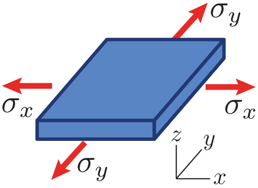
$$\tau_{xy} = \tau_{xz} = \tau_{yz} = 0$$

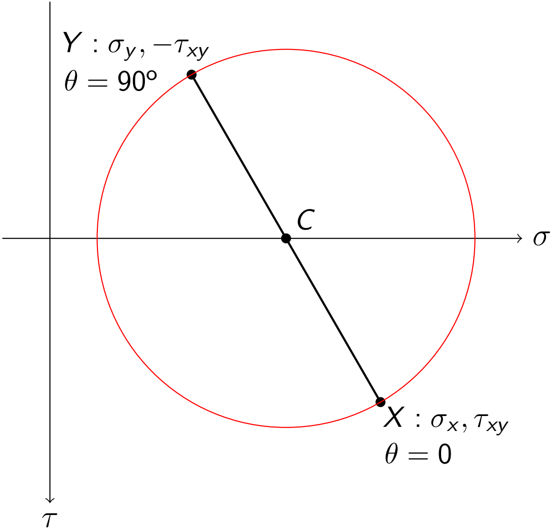
$$[\sigma_{ij}] = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix}$$



Cercle de Mohr 3D

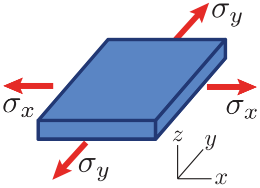
$$\boxed{\sigma_z = \tau_{zx} = \tau_{zy} = 0}$$

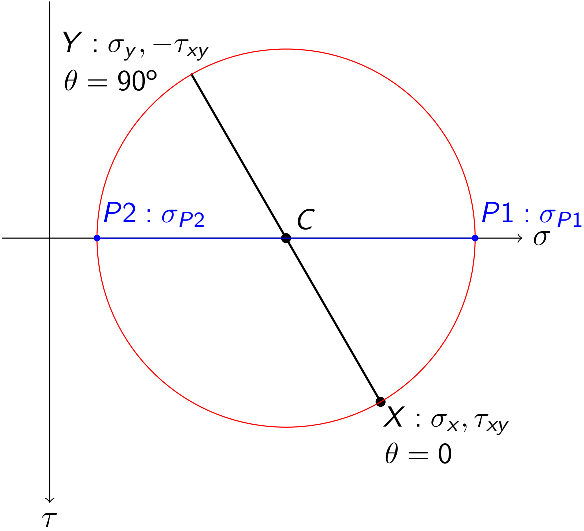




Cercle de Mohr 3D

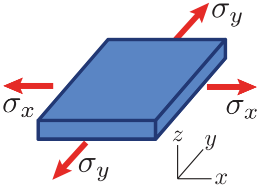
$$\boxed{\sigma_z = \tau_{zx} = \tau_{zy} = 0}$$

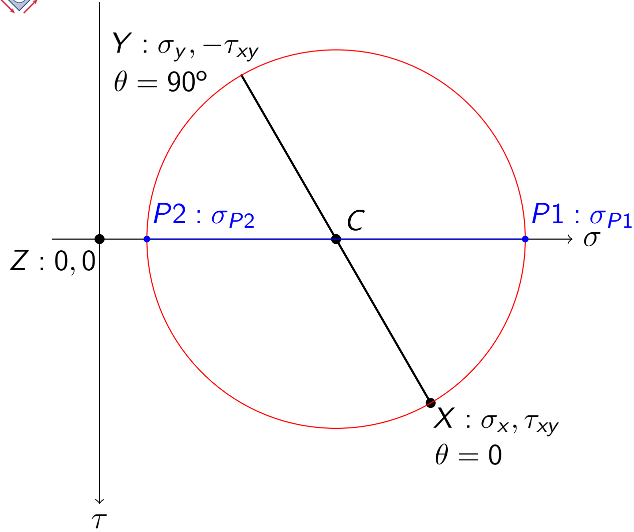




Cercle de Mohr 3D

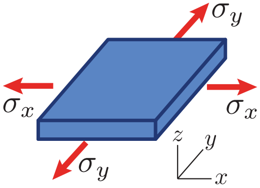
$$\boxed{\sigma_z = \tau_{zx} = \tau_{zy} = 0}$$

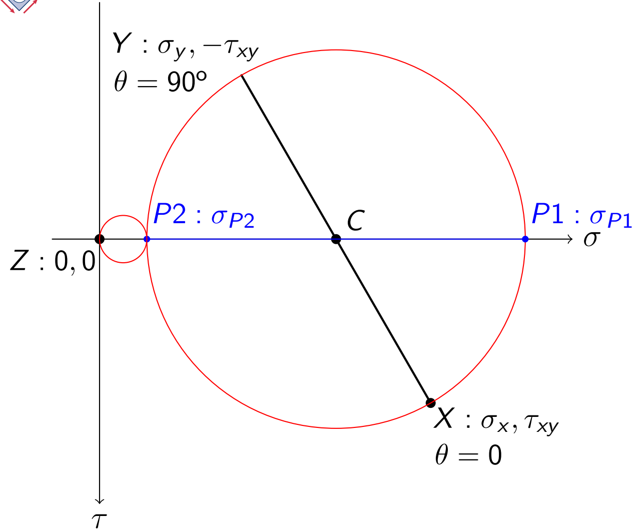




Cercle de Mohr 3D

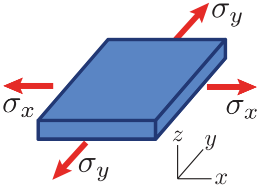
$$\boxed{\sigma_z = \tau_{zx} = \tau_{zy} = 0}$$

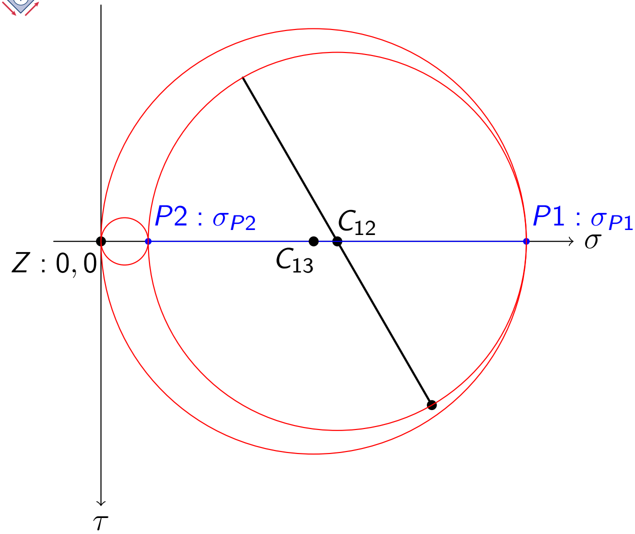




Cercle de Mohr 3D

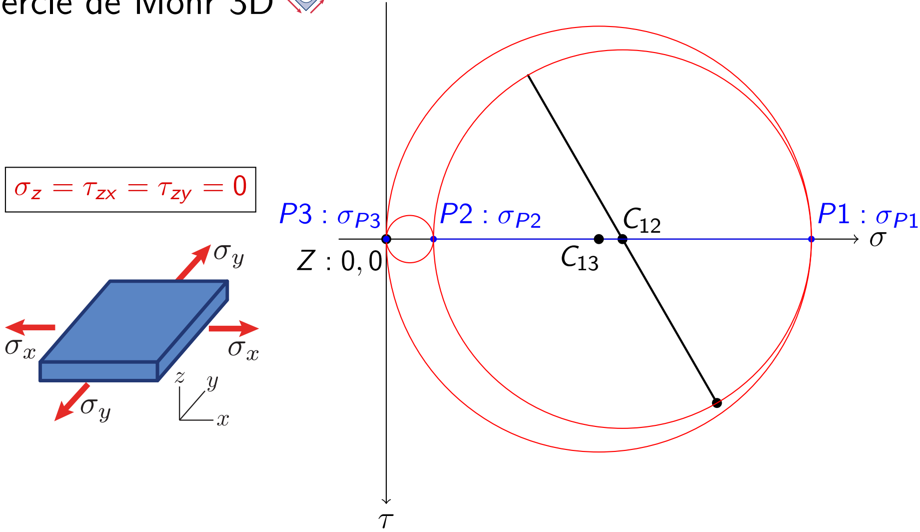
$$\boxed{\sigma_z = \tau_{zx} = \tau_{zy} = 0}$$





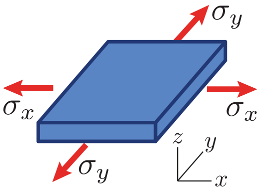
Cercle de Mohr 3D

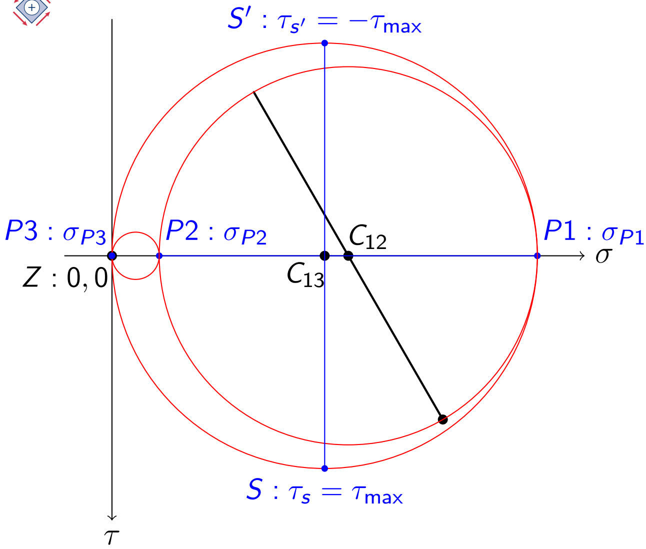
$$\boxed{\sigma_z = \tau_{zx} = \tau_{zy} = 0}$$



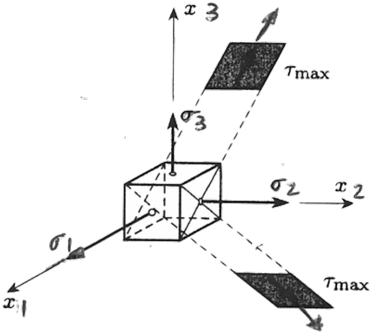
Cercle de Mohr 3D

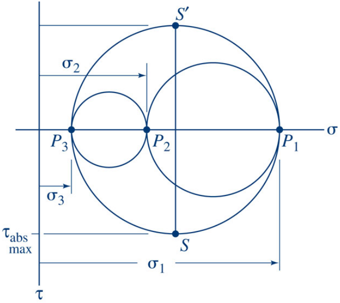
$$\boxed{\sigma_z = \tau_{zx} = \tau_{zy} = 0}$$





Cercle de Mohr 3D (cont.)



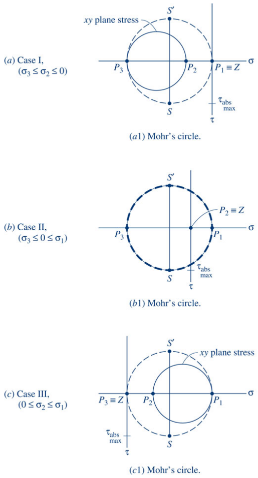


Pour $\sigma_1 \geq \sigma_2 \geq \sigma_3$

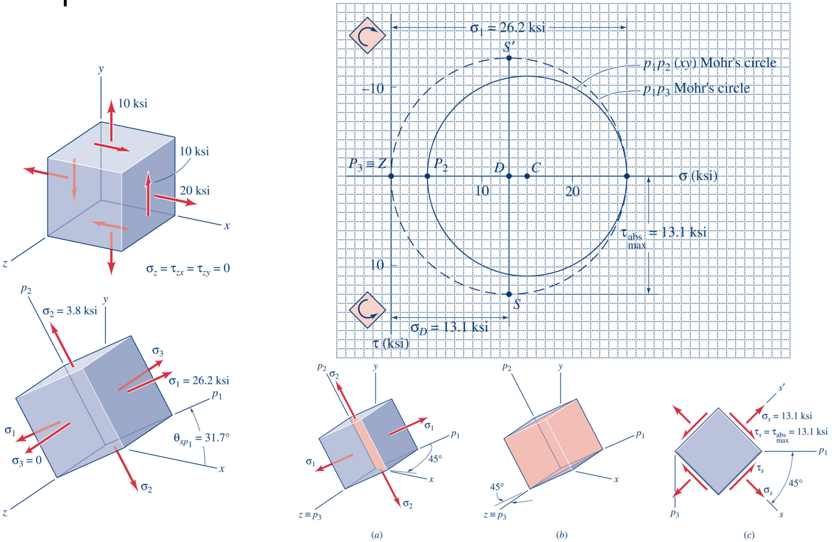
$$\begin{aligned} \sigma_{\text{max}} &= \sigma_1 \\ \sigma_{\text{min}} &= \sigma_3 \\ |\tau_{\text{max}}| &= 1/2(\sigma_1 - \sigma_3) \\ \sigma_S &= 1/2(\sigma_1 + \sigma_3) \end{aligned}$$

[Frey, 1998]

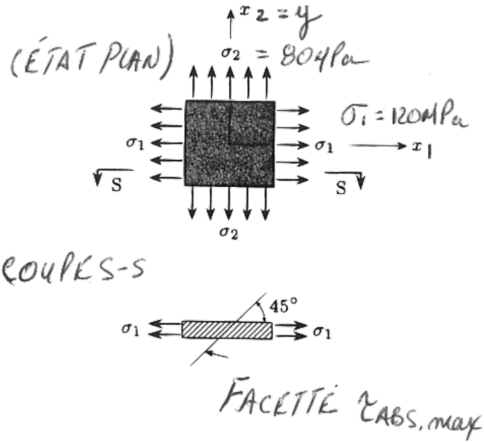
Cercle de Mohr 3D – contraintes principales



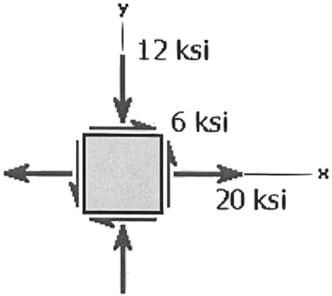
Exemple 7


y
10 ksi
10 ksi
20 ksi
x
z
$\sigma_z = \tau_{zx} = \tau_{zy} = 0$
p_2
$\sigma_2 = 3.8$ ksi
σ_3
$\sigma_1 = 26.2$ ksi
p_1
σ_1
$\sigma_3 = 0$
$\theta_{xp_1} = 31.7°$
σ_2
$\sigma_1 = 26.2$ ksi
S'
p_1p_2 (xy) Mohr's circle
p_1p_3 Mohr's circle
−10
$P_3 \equiv Z$
P_2
D
C
σ (ksi)
10
20
$\tau_{\substack{abs\\max}} = 13.1$ ksi
10
S
$\sigma_D = 13.1$ ksi
τ (ksi)
45°
$z \equiv p_3$
(a)
(b)
(c)
s'
$\sigma_s = 13.1$ ksi
$\tau_s = \tau_{\substack{abs\\max}} = 13.1$ ksi
τ_s
σ_s
p_3
s

Exemple 8

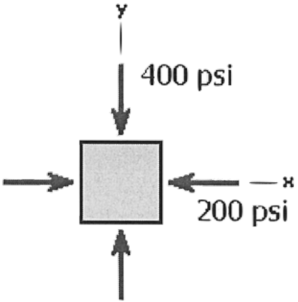

(ÉTAT PLAN)
$x_2 = y$
$\sigma_2 = 80$ MPa
$\sigma_1 = 120$ MPa
σ_1
σ_1
x_1
S
S
σ_2
COUPE S-S
45°
σ_1
σ_1
FACETTE $\tau_{ABS,max}$

Exemple 9



$$\sigma_z = 0$$

Exemple 10



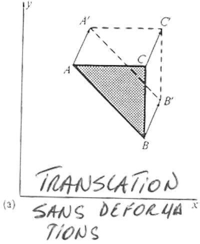
$\sigma_z = 0$

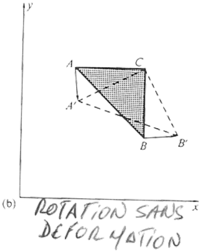
Corps déformables

① Mouvement rigide

(a) translation
(b) rotation

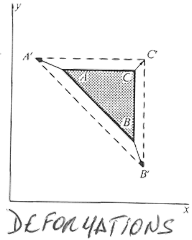
qui n'entraîne aucune déformation unitaire





② Mouvement relatif d'un point par rapport à un autre

- théorie de petits déplacements
- géométries initiale et finale sont à peu près équivalents



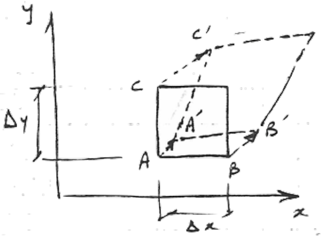
Déformations dans un plan

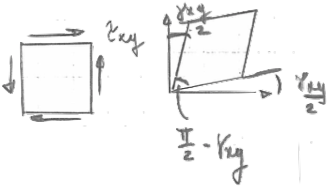
$$\varepsilon_x = \lim_{\Delta x \to 0} \frac{A'B' - AB}{AB}$$

$$\varepsilon_y = \lim_{\Delta y \to 0} \frac{A'C' - AC}{AC}$$

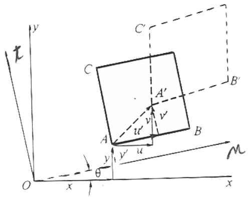
$$\gamma_{xy} = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \tan\left(B\widehat{A}C - B'\widehat{A'}C'\right)$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \tan\left(\frac{\pi}{2} - B'\widehat{A'}C'\right)$$





Déformations selon des plans arbitraires



Transformation des déformations

$$\epsilon_n = \left(\frac{\epsilon_x+\epsilon_y}{2}\right) + \left(\frac{\epsilon_x-\epsilon_y}{2}\right)\cos 2\theta + \left(\frac{\gamma_{xy}}{2}\right)\sin 2\theta$$

$$\epsilon_t = \left(\frac{\epsilon_x+\epsilon_y}{2}\right) - \left(\frac{\epsilon_x-\epsilon_y}{2}\right)\cos 2\theta - \left(\frac{\gamma_{xy}}{2}\right)\sin 2\theta$$

$$\frac{\gamma_{nt}}{2} = -\left(\frac{\epsilon_x-\epsilon_y}{2}\right)\sin 2\theta + \left(\frac{\gamma_{xy}}{2}\right)\cos 2\theta$$

$$\epsilon_n + \epsilon_t = \epsilon_x + \epsilon_y$$

Déformations principales (normales)

$$\epsilon_1 = \left(\frac{\epsilon_x+\epsilon_y}{2}\right) + \sqrt{\left(\frac{\epsilon_x-\epsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2} = \epsilon_{avg.} + R$$

$$\epsilon_2 = \left(\frac{\epsilon_x+\epsilon_y}{2}\right) - \underbrace{\sqrt{\left(\frac{\epsilon_x-\epsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}}_{R} = \epsilon_{avg.} - R$$

$$\tan 2\theta_{xp1} = \frac{\gamma_{xy}}{\epsilon_x - \epsilon_y}$$

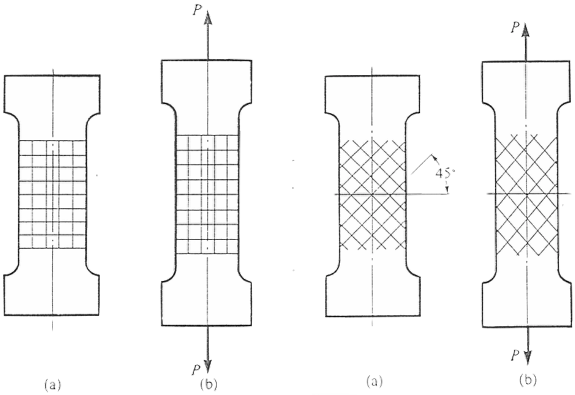
Déformations principales (cisaillement)

$$\gamma_{max} = -\gamma_{min} = \epsilon_1 - \epsilon_2$$

$$\epsilon_{S1} = \epsilon_{S2} = \frac{1}{2}(\epsilon_1 + \epsilon_2)$$

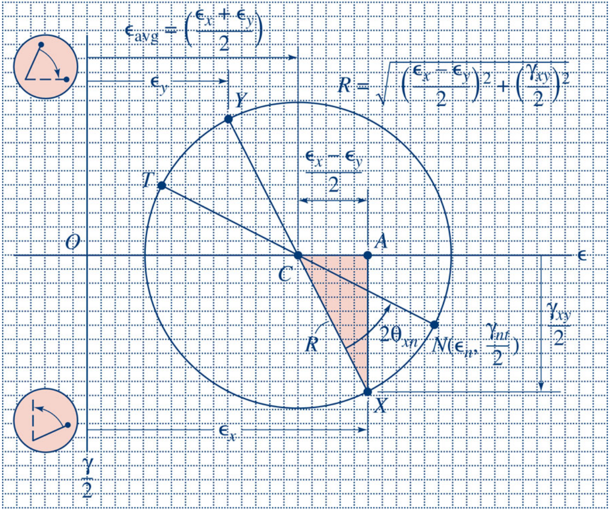
Déformations principales

Déformations principales : aucune distorsion en déformation ($\gamma_{xy} = 0$, i.e. les angles droits restent droits)

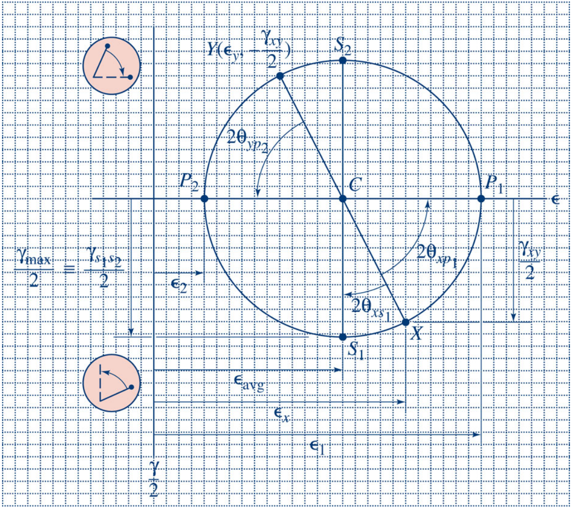


[Bazergui]

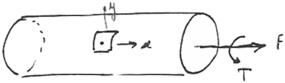
Cercle de Mohr – déformations

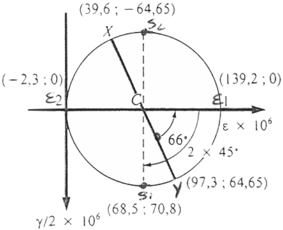


Cercle de Mohr - déformations principales



Déformations principales - exemple





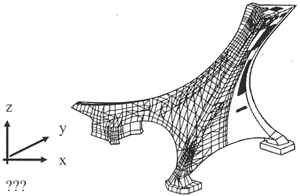
$$
\begin{aligned}
\epsilon_x &= 39.6 \times 10^{-6}\\
\epsilon_y &= 97.3 \times 10^{-6}\\
\gamma_{xy} &= -129.3 \times 10^{-6}
\end{aligned}
$$

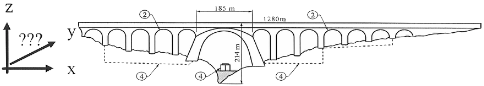
$\epsilon_1 = ?,\ \epsilon_2 = ?,\ \gamma_{max} = ?, \theta_{xp1} = ?, \theta_{xs1} = ?$

Déformations principales

$$
\begin{aligned}
\epsilon_{1,2} &= \left(\frac{\epsilon_x+\epsilon_y}{2}\right) \pm \sqrt{\left(\frac{\epsilon_x-\epsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}\\
\gamma_{\text{max}} &= -\gamma_{\text{min}} = \epsilon_1 - \epsilon_2
\end{aligned}
$$

Comment connaitre les contraintes principales en 2D-3D?





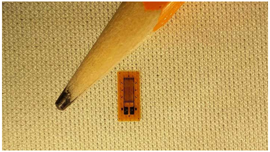
$$\underbrace{\{\sigma_1, \sigma_2\}}_{\sigma \text{ principales}} \leftarrow \underbrace{\{\epsilon_1, \epsilon_2\}}_{\epsilon \text{ principales}} \leftarrow \underbrace{\{\epsilon_x, \epsilon_y, \gamma_{xy}\}}_{\epsilon \text{ cartésiennes}} \leftarrow \underbrace{\{\epsilon_a, \epsilon_b, \epsilon_c\}}_{\epsilon \text{ mesurées}}$$

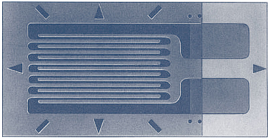
Requiert la mesure des déformations axiales ϵ selon trois directions: a, b, c

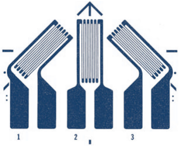
Comment mesure-t-on les déformations en 2D-3D?

Jauges de déformation axiales

$$\boxed{\epsilon_n \propto \text{résistance électrique}}$$



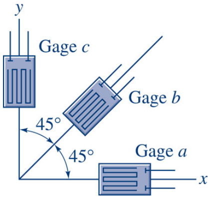




Rosettes 45° – transformation des déformations

Transformation des déformations

$$\epsilon_n = \left(\frac{\epsilon_x + \epsilon_y}{2}\right) + \left(\frac{\epsilon_x - \epsilon_y}{2}\right)\cos 2\theta + \left(\frac{\gamma_{xy}}{2}\right)\sin 2\theta$$



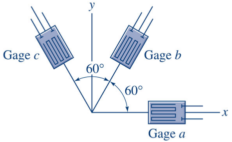
Pour une rosette à 45°

$$\begin{aligned}
\epsilon_x &= \epsilon_a \\
\epsilon_y &= \epsilon_c \\
\epsilon_b &= \left(\frac{\epsilon_x+\epsilon_y}{2}\right) + \left(\frac{\epsilon_x-\epsilon_y}{2}\right)\overbrace{\cos 2\theta}^{=0} + \left(\frac{\gamma_{xy}}{2}\right)\overbrace{\sin 2\theta}^{=1} \\
&= \left(\frac{\epsilon_x+\epsilon_y}{2}\right) + \left(\frac{\gamma_{xy}}{2}\right) \\
&\rightarrow \gamma_{xy} = 2\epsilon_b - \epsilon_x - \epsilon_y
\end{aligned}$$

Rosettes 60° – transformation des déformations

Transformation des déformations

$$\epsilon_n = \left(\frac{\epsilon_x + \epsilon_y}{2}\right) + \left(\frac{\epsilon_x - \epsilon_y}{2}\right)\cos 2\theta + \left(\frac{\gamma_{xy}}{2}\right)\sin 2\theta$$



Pour une rosette à 60°

$$\begin{aligned}
\epsilon_x &= \epsilon_a \\
\epsilon_y &= \frac{1}{3}(2\epsilon_b + 2\epsilon_c - \epsilon_a) \\
\gamma_{xy} &= \frac{2}{\sqrt{3}}(\epsilon_b - \epsilon_c)
\end{aligned}$$

Déformations et contraintes principales

Déformations principales (normales)

$$\epsilon_1 = \left(\frac{\epsilon_x+\epsilon_y}{2}\right) + \sqrt{\left(\frac{\epsilon_x-\epsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$

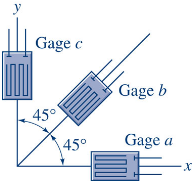
$$\epsilon_2 = \left(\frac{\epsilon_x+\epsilon_y}{2}\right) - \sqrt{\left(\frac{\epsilon_x-\epsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$

Contraintes principales (contraintes planes, $\sigma_3 = 0$)

$$\sigma_1 = \frac{E}{1-\nu^2}(\epsilon_1 + \nu\epsilon_2)$$

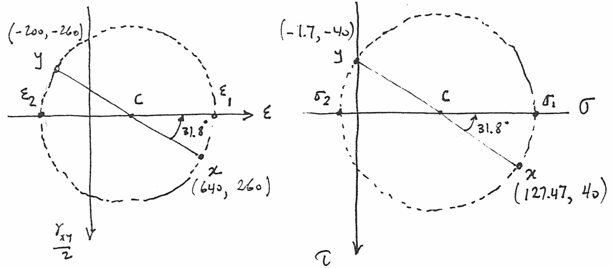
$$\sigma_2 = \frac{E}{1-\nu^2}(\epsilon_2 + \nu\epsilon_1)$$

Contraintes principales & rosette – exemple



$$\begin{aligned}\epsilon_a &= 640\times10^{-6}\\ \epsilon_b &= 480\times10^{-6}\\ \epsilon_c &= -200\times10^{-6}\\ E &= 200\ GPa\\ \nu &= 0.3\end{aligned}$$

$\sigma_1 = ?,\ \sigma_2 = ?,\ \theta_{xp1}$

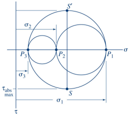


Résumé – Module #8b

But : méthode graphique pour calculer

$$\underbrace{\sigma_n, \tau_{nt}}_{\text{contraintes normales et tangentielles}} \;,\; \underbrace{\sigma_1, \sigma_2, \theta_p}_{\text{contraintes principales}} \;,\; \underbrace{\tau_{s1}, \theta_s}_{\text{cisaillement maximal}}$$

Cercle de Mohr : $(\sigma_n - \sigma_{avg.})^2 + \tau_{nt}^2 = R^2$



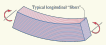
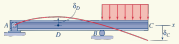
σ_p à partir des mesures de déformations axiales :

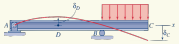
$$\underbrace{\{\sigma_1, \sigma_2\}}_{\sigma\text{ principales}} \leftarrow \underbrace{\{\epsilon_1, \epsilon_2\}}_{\epsilon\text{ principales}} \leftarrow \underbrace{\{\epsilon_x, \epsilon_y, \gamma_{xy}\}}_{\epsilon\text{ cartésiennes}} \leftarrow \underbrace{\{\epsilon_a, \epsilon_b, \epsilon_c\}}_{\epsilon\text{ mesurées}}$$

Organisation de la matière

1 *Statique*
- Équilibre des forces et moments
- Diagrammes de corps libres 
- **5** Diagramme des efforts, $N(x)$, $V(x)$, $M(x)$

2 *Matériau*
- Contraintes & déformations
- Loi de Hooke, Poisson & St-Venant  

Chargements
- **3** Efforts axiaux 
- **4** Torsion 
- **6*a*** Flexion 
- **6*b*** Cisaillement 
- **7** Déflexion 
- **9** Pression & chargements combinés  

États limites
- **7** Déflexion 
- **8** Contraintes 2D-3D 
- **10** Lois constitutives & critères de rupture