Module #8a
Transformation des contraintes et des déformations 2D-3D : formulation analytique

(CIV1150 - Résistance des matériaux)

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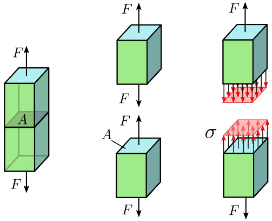
Sections 8.1-8.5 – R. Craig (2011)
Mechanics of Materials, 3rd Edition
John Wiley & Sons.

P. Léger (2006)
Notes de cours: Chapitre 8, §8.1–8.6
Polytechnique Montréal.

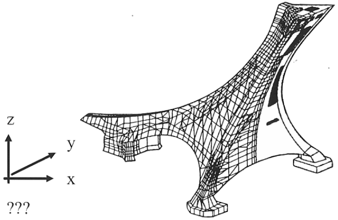
Mise en contexte

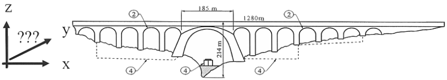
Contrainte dans la barre?



$$\sigma = \frac{F}{A}$$

Contrainte dans la structure?

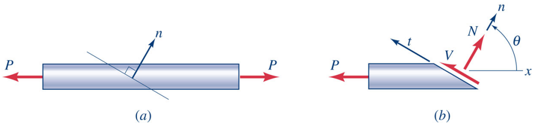




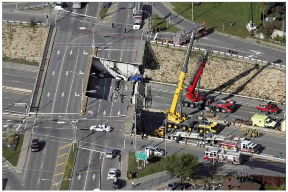
Requiert l'analyse des contraintes principales en 2D-3D

Objectifs

Chapitre 2: Transformations des contraintes 1D

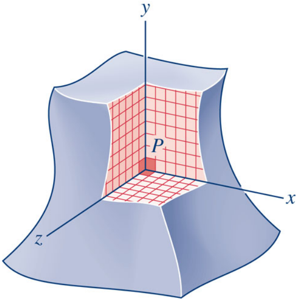


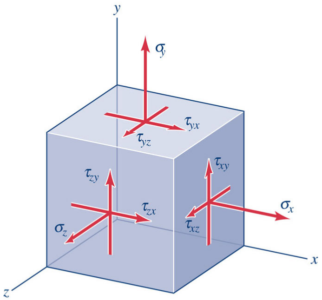
Chapitre 8, partie a : formulation analytique 2D-3D
Chapitre 8, partie b : approche graphique – cercle de Mohr



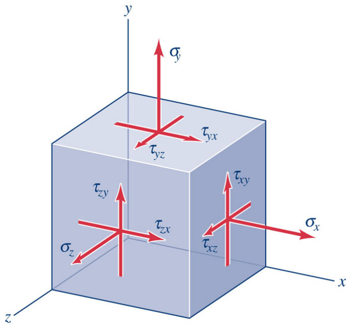
Qu'est-ce qui explique l'orientation de la fissure?

État des contraintes 3D





Tenseur de contraintes 3D



Tenseur de contraintes

i: Plan normal, j: Direction

$$[\sigma_{ij}] = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix}$$

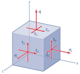
Afin de satisfaire les éq. d'équilibre:

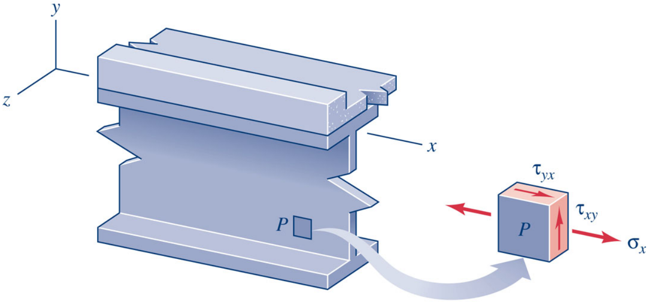
$$\tau_{xy} = \tau_{yx}, \; \tau_{xz} = \tau_{zx}, \; \tau_{yz} = \tau_{zy}$$

Dans plusieurs situations, l'**état de contraintes** peut-être réduit à une représentation **2D** $\rightarrow$ **contraintes planes**

Chargement 2D - contraintes planes

Contraintes planes: $\sigma_z = \tau_{xz} = \tau_{yz} = 0$

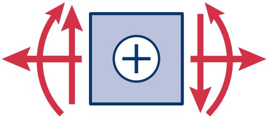




(*a*) A portion of a bridge girder.

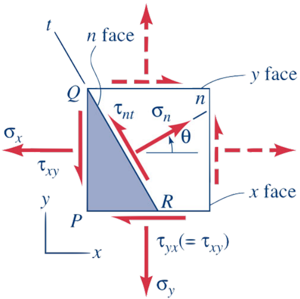
(*b*) A web element, with stresses shown.

Convention de signes - ⚠ **efforts internes**



Convention de signes pour les efforts internes

Convention de signes - ⚠ **contraintes** *xy*



Contraintes **normales**:

- traction (+)
- compression (-)

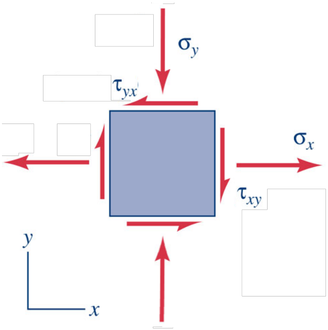
Contraintes de **cisaillements**:

- signe +: selon la direction de t positive

  $\tau_{nt} \rightarrow$ n: plan normal, t: plan tangent

- signe +: rotation **anti-horaire** 

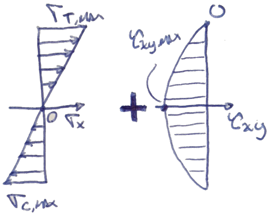
Exercice - convention de signes

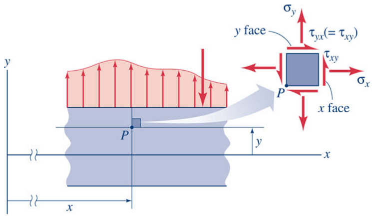


Positif ou négatif?

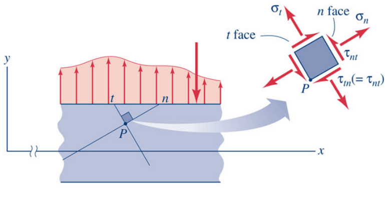
$$\sigma_x > 0,\ \sigma_y < 0,\ \tau_{xy} < 0,\ \tau_{yx} > 0$$

Choix du système d'axe





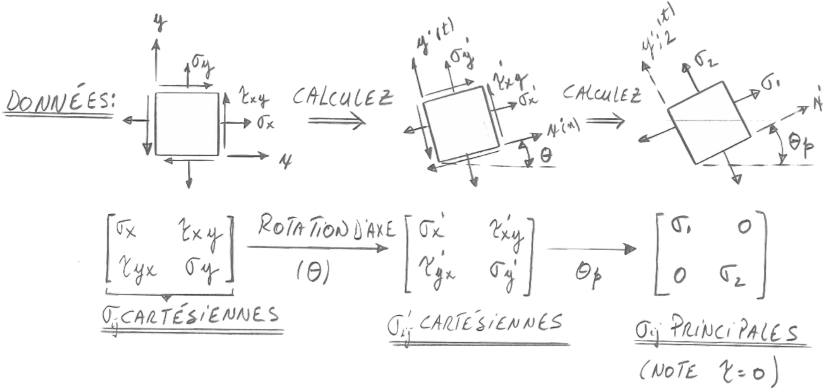
(a) The stress state at point P, referred to x and y axes.



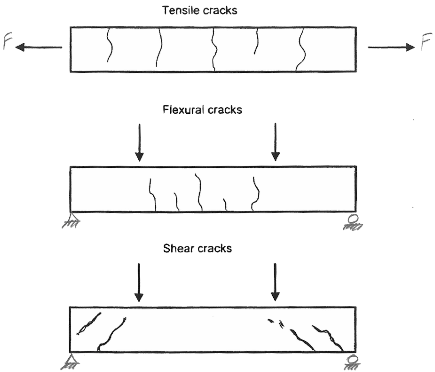
(b) The stress state at point P, referred to n and t axes.

Contraintes principales

Une des orientations clefs pour le système d'axe est celle correspondant aux **contraintes principales (i.e. $\tau_{xy} = 0$)**

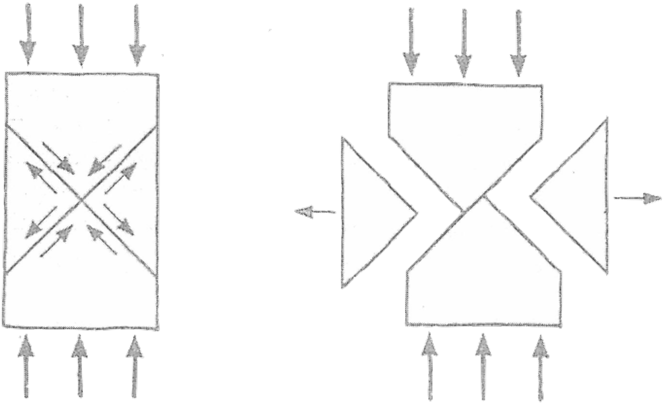


Fissuration des éléments en béton

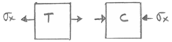
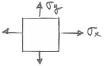
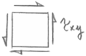
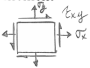


[ICOLD, 1997]

Critère de rupture béton



Contraintes principales – cas communs 2D

		σ_{MAX} ; τ_{MAX} (PRINCIPALE)
1- Contraintes uni-axiales (tension ou compression) 	$\begin{bmatrix} \sigma_x & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$	$\sigma_1 = \sigma_x$ $\tau_{max} = \frac{\sigma_x}{2}$ (à $\theta_p = 45°$)
2- Contraintes bi-axiales 	$\begin{bmatrix} \sigma_x & 0 & 0 \\ 0 & \sigma_y & 0 \\ 0 & 0 & 0 \end{bmatrix}$	$\sigma_1 = \sigma_x$, $\sigma_2 = \sigma_y$ τ_{max} = LE PLUS GRAND DE $\frac{1}{2}(\sigma_x - \sigma_y)$, $\frac{1}{2}\sigma_x$, $\frac{1}{2}\sigma_y$
3- Cisaillement pur (torsion) 	$\begin{bmatrix} 0 & \tau_{xy} & 0 \\ \tau_{yx} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$	$\sigma_1 = +\tau_{xy}$ $\sigma_2 = -\tau_{xy}$ $\tau_{max} = \tau_{xy}$
4- Contraintes biaxiales et cisaillement 	$\begin{bmatrix} \sigma_x & \tau_{xy} & 0 \\ \tau_{yx} & \sigma_y & 0 \\ 0 & 0 & 0 \end{bmatrix}$	$\sigma_1, \sigma_2 = \frac{1}{2}(\sigma_x + \sigma_y) \pm \sqrt{\frac{(\sigma_x - \sigma_y)^2}{4} + \tau_{xy}^2}$ $\tau_{MAX} = \sqrt{\frac{(\sigma_x - \sigma_y)^2}{4} + \tau_{xy}^2}$

Comment obtenir ces résultats?

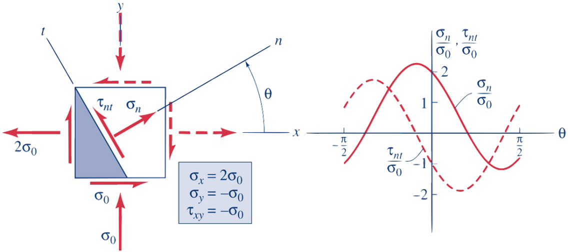
Procédure pour calculer les σ principales et leur orientation

1 – Transformation des contraintes (forces) dans le système d'axe global $x - y$ au système d'axe local $n - t$

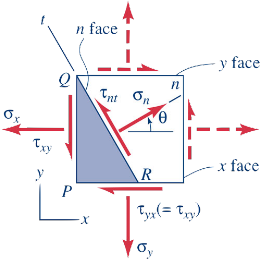
$$(x, y) \rightarrow (n, t, \theta)$$

2 – Trouver l'angle θ_p

Option a : maximisation $\frac{d\sigma_n}{d\theta} = 0$, **Option b** : valeurs propres

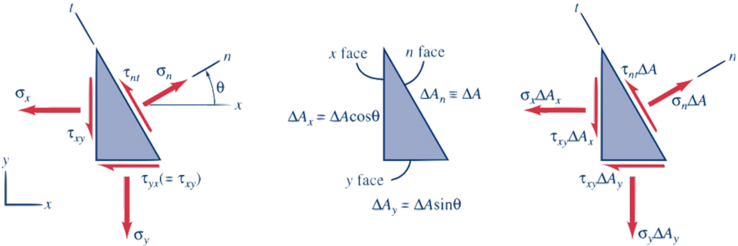


Transformation des contraintes – Équilibre des forces



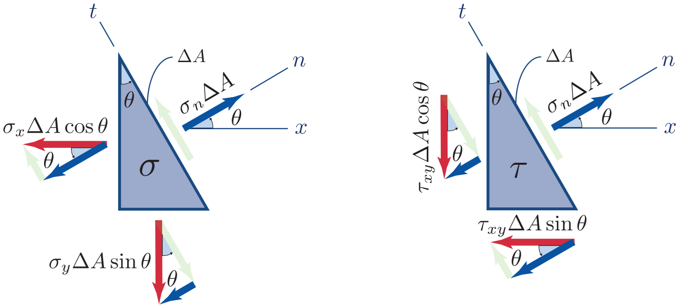
$$\overset{+}{\nearrow} \sum F_n = 0$$

$$\overset{+}{\nwarrow} \sum F_t = 0$$



(*a*) A triangular stress element. (*b*) The areas of the respective faces. (*c*) The free-body diagram.

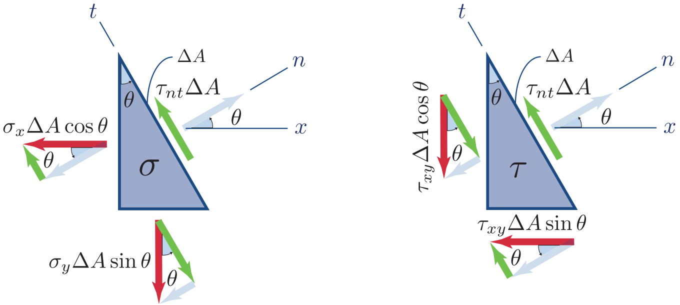
Transformation des contraintes : $\overset{+}{\nearrow} \sum F_n = 0$



$$\overset{+}{\nearrow} \sum F_n = 0 \quad = \quad \sigma_n \Delta A - \sigma_x(\Delta A \cos\theta)\cos\theta - \sigma_y(\Delta A \sin\theta)\sin\theta - \tau_{xy}(\Delta A \cos\theta)\sin\theta - \tau_{xy}(\Delta A \sin\theta)\cos\theta$$

$$\boxed{\sigma_n = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + \tau_{xy}(2\sin\theta\cos\theta)}$$

Transformation des contraintes : $\overset{+}{\nwarrow} \sum F_t = 0$



$$
\begin{aligned}
\overset{+}{\nwarrow} \sum F_t = 0 \quad = \quad & \tau_{nt}\Delta A - \sigma_x(\Delta A \cos\theta)\sin\theta - \sigma_y(\Delta A \sin\theta)\cos\theta \\
& - \tau_{xy}(\Delta A \cos\theta)\cos\theta - \tau_{xy}(\Delta A \sin\theta)\cos\theta
\end{aligned}
$$

$$
\boxed{\tau_{nt} = -(\sigma_x - \sigma_y)\cos\theta\sin\theta + \tau_{xy}(\cos^2\theta - \sin^2\theta)}
$$

Transformation des contraintes

$$\boxed{\begin{aligned} \sigma_n &= \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + \tau_{xy}(2\sin\theta\cos\theta) \\ \tau_{nt} &= -(\sigma_x - \sigma_y)\cos\theta\sin\theta + \tau_{xy}(\cos^2\theta - \sin^2\theta) \end{aligned}}$$

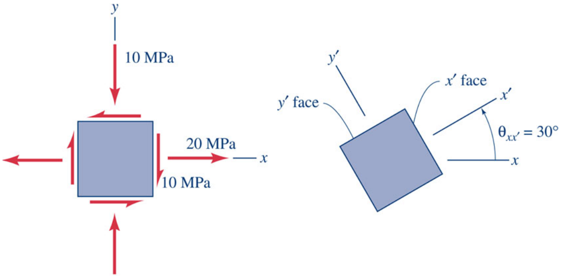
$\Downarrow$ identités géométriques $\Downarrow$

Formules rotation du système d'axe

$$\begin{aligned} \sigma_n &= \left(\frac{\sigma_x+\sigma_y}{2}\right) + \left(\frac{\sigma_x-\sigma_y}{2}\right)\cos 2\theta + \tau_{xy}\sin 2\theta \\ \tau_{nt} &= -\left(\frac{\sigma_x-\sigma_y}{2}\right)\sin 2\theta + \tau_{xy}\cos 2\theta \end{aligned}$$

Note : angle θ en $^\circ$, $+$*antihoraire*

Rotation d'axes – exemple

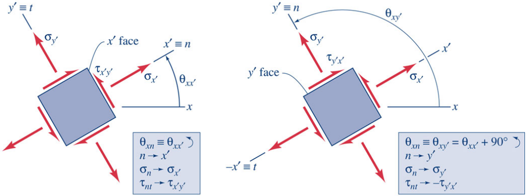


$$\sigma_{x'} = ?,\ \ \tau_{x'y'} = ?$$

Vérifier la validité des équations

$$\boxed{\begin{aligned}\sigma_n &= \left(\frac{\sigma_x+\sigma_y}{2}\right) + \left(\frac{\sigma_x-\sigma_y}{2}\right)\cos 2\theta + \tau_{xy}\sin 2\theta \\ \tau_{nt} &= -\left(\frac{\sigma_x-\sigma_y}{2}\right)\sin 2\theta + \tau_{xy}\cos 2\theta\end{aligned}}$$

Rotation d'axes - cas général



$$\boxed{\sigma_{x'} + \sigma_{y'} = \sigma_x + \sigma_y}$$

La somme des contraintes sur deux faces orthogonales est **indépendante de θ**

Formules rotation du système d'axe

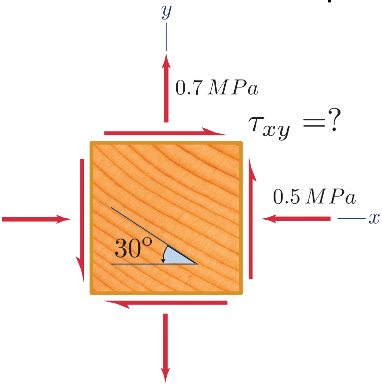
$$\sigma_{x'} \equiv \sigma_n(\theta_{xx'}) = \left(\frac{\sigma_x+\sigma_y}{2}\right) + \left(\frac{\sigma_x-\sigma_y}{2}\right)\cos 2\theta_{xx'} + \tau_{xy}\sin 2\theta_{xx'}$$

$$\sigma_{y'} \equiv \sigma_n(\theta_{xx'} + 90°) = \left(\frac{\sigma_x+\sigma_y}{2}\right) - \left(\frac{\sigma_x-\sigma_y}{2}\right)\cos 2\theta_{xx'} - \tau_{xy}\sin 2\theta_{xx'}$$

$$\tau_{x'y'} \equiv \tau_{nt}(\theta_{xx'}) = -\left(\frac{\sigma_x-\sigma_y}{2}\right)\sin 2\theta_{xx'} + \tau_{xy}\cos 2\theta_{xx'}$$

$$\tau_{x'y'} = \tau_{y'x'}$$

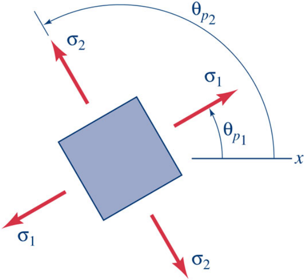
Rotation d'axes – exemple



Question:
Si $\tau_{adm.} = \pm 0.4\,MPa$ dans la direction des fibres, calculez le cisaillement admissible τ_{xy} tel que $|\tau_{x'y'}| < \tau_{adm.}$

$$\boxed{\tau_{x'y'} = -\left(\frac{\sigma_x - \sigma_y}{2}\right)\sin 2\theta_{xx'} + \tau_{xy}\cos 2\theta_{xx'}}$$

Définition – contraintes principales



θ_{p1} et θ_{p2} décrivent les plans où les contraintes sont respectivement maximales (σ_1) et minimales (σ_2)

$$\sigma_1 \geq \sigma_2$$

$$\theta_{p2} = \theta_{p1} \pm 90^o$$

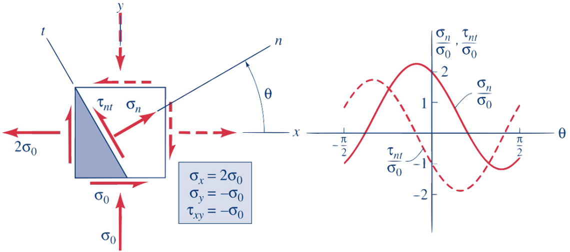
Procédure pour calculer les σ principales et leur orientation

△ 1 – Transformation des contraintes (forces) dans le système d'axe global $x-y$ au système d'axe local $n-t$

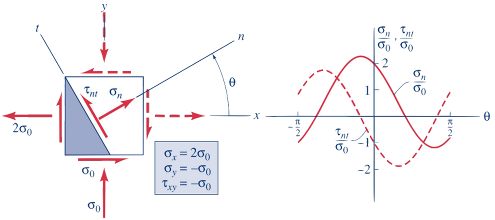
$$(x, y) \rightarrow (n, t, \theta)$$

2 – Trouver l'angle θ_p

Option a : maximisation $\frac{d\sigma_n}{d\theta} = 0$, **Option b** : valeurs propres



Maximisation – Contraintes principales



Contraintes principales: σ_1 et σ_2 sont obtenues pour l'angle θ_p tel que

$$\frac{d\sigma_n}{d\theta} = 0$$

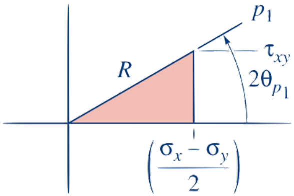
$$\sigma_n = \left(\frac{\sigma_x + \sigma_y}{2}\right) + \left(\frac{\sigma_x - \sigma_y}{2}\right)\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\frac{d\sigma_n}{d\theta} = -(\sigma_x - \sigma_y)\sin 2\theta_p + 2\tau_{xy}\cos 2\theta_p = 0$$

$$\boxed{\tan 2\theta_p = \frac{\tau_{xy}}{\left(\frac{\sigma_x - \sigma_y}{2}\right)}}$$

Représentation graphique – Contraintes principales

$$\tan 2\theta_p = \frac{\tau_{xy}}{\left(\frac{\sigma_x - \sigma_y}{2}\right)}$$

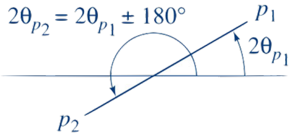


Relations géométriques

$$R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\sin 2\theta_{p1} = \frac{\tau_{xy}}{R}, \quad \cos 2\theta_{p1} = \frac{\left(\frac{\sigma_x - \sigma_y}{2}\right)}{R}$$

$$\sin 2\theta_{p2} = \frac{-\tau_{xy}}{R}, \quad \cos 2\theta_{p2} = \frac{-\left(\frac{\sigma_x - \sigma_y}{2}\right)}{R}$$



Rappel: $\sigma_n = \left(\frac{\sigma_x + \sigma_y}{2}\right) + \left(\frac{\sigma_x - \sigma_y}{2}\right) \cos 2\theta + \tau_{xy} \sin 2\theta$

Contraintes principales

$$R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

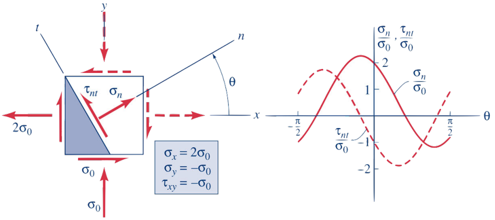
Formules contraintes principales

$$\begin{aligned}
\sigma_{p1} \equiv \sigma_n(\theta_{p1}) &= \left(\frac{\sigma_x+\sigma_y}{2}\right) + \left(\frac{\sigma_x-\sigma_y}{2}\right)\frac{\left(\frac{\sigma_x-\sigma_y}{2}\right)}{R} + \tau_{xy}\frac{\tau_{xy}}{R} \\
&= \left(\frac{\sigma_x+\sigma_y}{2}\right) + \frac{R^2}{R} \\
&= \sigma_{\text{avg.}} + R \\
\sigma_{p2} \equiv \sigma_n(\theta_{p2}) &= \sigma_{\text{avg.}} - R \\
\tau_{nt}(\theta_{p1}) = \tau_{nt}(\theta_{p2}) &= \mathbf{0} ⚠
\end{aligned}$$

Contraintes principales

$$\sigma_{p1,p2} = \underbrace{\sigma_{\text{avg.}}}_{\substack{\text{Composante}\\ \text{hydrostatique}}} \pm \underbrace{R}_{\substack{\text{Composante}\\ \text{déviatorique}}}$$

Maximisation – Cisaillement maximal



Cisaillement maximal:

τ_{s1} et τ_{s2} sont obtenues pour l'angle θ_s tel que

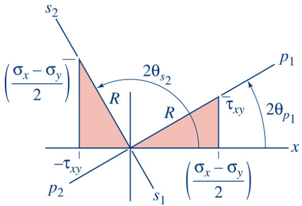
$$\frac{d\tau_{nt}}{d\theta} = 0$$

$$\frac{d\tau_{nt}}{d\theta} = -\left(\sigma_x - \sigma_y\right)\cos 2\theta_s - 2\tau_{xy}\sin 2\theta_p = 0$$

$$\boxed{\tan 2\theta_p = -\frac{\left(\frac{\sigma_x - \sigma_y}{2}\right)}{\tau_{xy}}}$$

Représentation graphique – Cisaillement maximal

$$\tan 2\theta_s = -\frac{\tau_{xy}}{\left(\frac{\sigma_x-\sigma_y}{2}\right)}$$



Relations géométriques

$$R = \sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\sin 2\theta_{s1} = \frac{-\left(\frac{\sigma_x-\sigma_y}{2}\right)}{R}, \quad \cos 2\theta_{s1} = \frac{\tau_{xy}}{R}$$

$$\sin 2\theta_{s2} = \frac{\left(\frac{\sigma_x-\sigma_y}{2}\right)}{R}, \quad \cos 2\theta_{s2} = \frac{-\tau_{xy}}{R}$$

Cisaillement maximal

$$\theta_s = \theta_p \pm 45^\circ$$

$$\tau_{s1} \equiv \tau_{nt}(\theta_{s1}) = R = \tau_{\max}$$

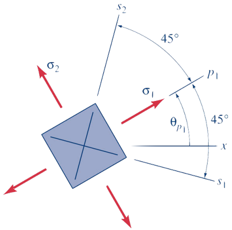
$$\tau_{s2} \equiv \tau_{nt}(\theta_{s2}) = -R = -\tau_{\max}$$

Résumé – calcul des contraintes et cisaillement maximaux

Contraintes principales

$$\sin 2\theta_{p1} = \frac{\tau_{xy}}{R}, \quad \theta_{p2} = \theta_{p1} + 90^\circ$$

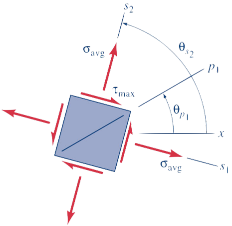
$$\begin{aligned} \sigma_{p1} \equiv \sigma_n(\theta_{p1}) &= \sigma_{\text{avg.}} + R \\ \sigma_{p2} \equiv \sigma_n(\theta_{p2}) &= \sigma_{\text{avg.}} - R \\ \tau_{nt}(\theta_{p1}) = \tau_{nt}(\theta_{p2}) &= 0 \; ⚠ \end{aligned}$$



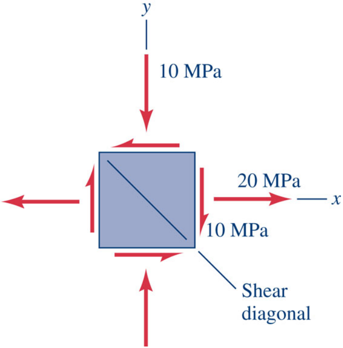
Cisaillement maximal

$$\cos 2\theta_{s1} = \frac{\tau_{xy}}{R}, \quad \theta_{s2} = \theta_{s1} + 90^\circ$$

$$\begin{aligned} \tau_{s1} \equiv \tau_{nt}(\theta_{s1}) &= R = \tau_{\max} \\ \tau_{s2} \equiv \tau_{nt}(\theta_{s2}) &= -R = -\tau_{\max} \\ \sigma_n(\theta_{s1}) = \sigma_n(\theta_{s2}) &= \sigma_{\text{avg.}} \; ⚠ \end{aligned}$$



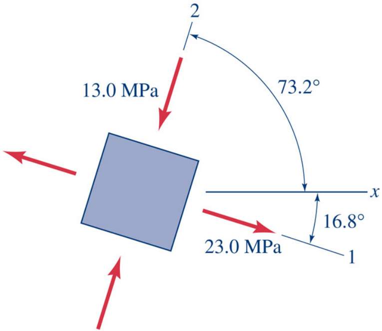
Contraintes principales – exemple

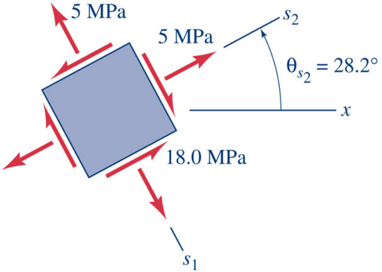


Question:
déterminer l'orientation $(\theta_{p1}, \theta_{s1})$ et les valeurs des contraintes

a) normales maximales $(\sigma_{p1}, \sigma_{p2})$
b) de cisaillement maximales $(\tau_{\max})$

Contraintes principales – exemple (cont.)





Champ de contraintes

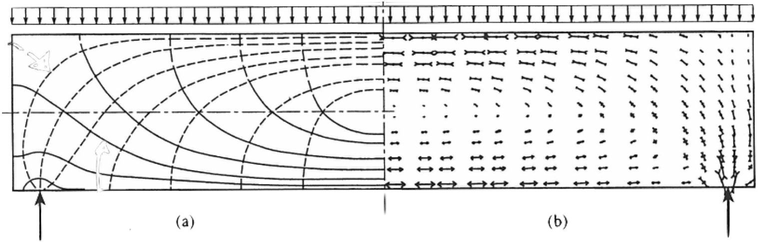
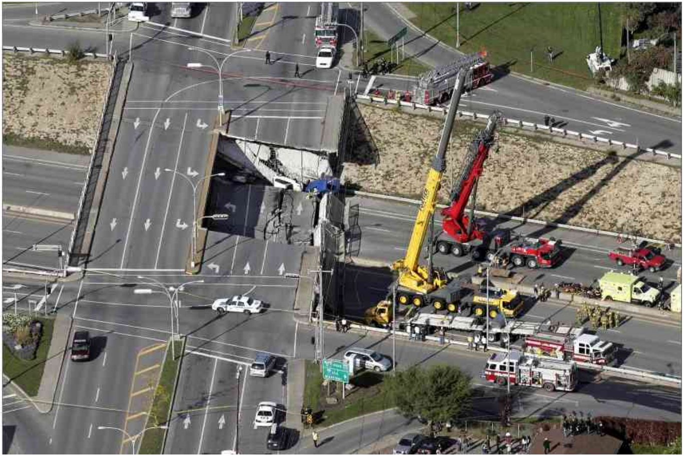


Fig. 2.21 Trajectoires des contraintes principales d'une poutre simple avec charge uniforme : (a) trajectoires (——— traction ; – – – compression) ; (b) croix.

[Frey, 1998]

Effondrement du Viaduc de la Concorde



[Bruno Massicotte]

Résumé – Module #8a

But : Calculer l'orientation et la valeur des contraintes principales

Formules rotation du système d'axe

$$
\begin{aligned}
\sigma_{x'} &\equiv \sigma_n(\theta_{xx'}) = \left(\frac{\sigma_x+\sigma_y}{2}\right) + \left(\frac{\sigma_x-\sigma_y}{2}\right)\cos 2\theta_{xx'} + \tau_{xy}\sin 2\theta_{xx'} \\
\sigma_{y'} &\equiv \sigma_n(\theta_{xx'}+90^\circ) = \left(\frac{\sigma_x+\sigma_y}{2}\right) - \left(\frac{\sigma_x-\sigma_y}{2}\right)\cos 2\theta_{xx'} - \tau_{xy}\sin 2\theta_{xx'} \\
\tau_{x'y'} &\equiv \tau_{nt}(\theta_{xx'}) = -\left(\frac{\sigma_x-\sigma_y}{2}\right)\sin 2\theta_{xx'} + \tau_{xy}\cos 2\theta_{xx'} \\
\tau_{x'y'} &= \tau_{y'x'}
\end{aligned}
$$

Contraintes principales

$$\sin 2\theta_{p1} = \frac{\tau_{xy}}{R}, \quad \theta_{p2} = \theta_{p1} + 90^\circ$$

$$
\begin{aligned}
\sigma_{p1} \equiv \sigma_n(\theta_{p1}) &= \sigma_{\text{avg.}} + R \\
\sigma_{p2} \equiv \sigma_n(\theta_{p2}) &= \sigma_{\text{avg.}} - R \\
\tau_{nt}(\theta_{p1}) = \tau_{nt}(\theta_{p2}) &= 0
\end{aligned}
$$

Cisaillement maximal

$$\cos 2\theta_{s1} = \frac{\tau_{xy}}{R}, \quad \theta_{s2} = \theta_{s1} + 90^\circ$$

$$
\begin{aligned}
\tau_{s1} \equiv \tau_{nt}(\theta_{s1}) &= R = \tau_{\max} \\
\tau_{s2} \equiv \tau_{nt}(\theta_{s2}) &= -R = -\tau_{\max} \\
\sigma_n(\theta_{s1}) = \sigma_n(\theta_{s2}) &= \sigma_{\text{avg.}}
\end{aligned}
$$

8b–Cercle de Mohr