

Module #6a

Contraintes de flexion dans les poutres

(CIV1150 - Résistance des matériaux)

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§6.1-6.4, 6.7 – R. Craig (2011)
Mechanics of Materials, 3rd Edition
John Wiley & Sons.

P. Léger (2006)
Notes de cours: Chapitre 6, §6.1–6.6
Polytechnique Montréal.

Objectifs

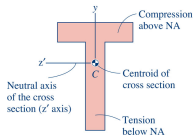
Chapitre 5: Diagrammes des efforts internes

- ▶ Moments fléchissants – $M(x)$
- ▶ Efforts tranchants – $V(x)$

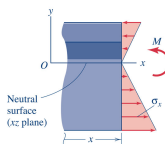
Chapitre 6, partie a: Contraintes de flexion dans les poutres

Chapitre 6, partie b: Contraintes de cisaillement dans les poutres

Connaissant $M(x)$, quelle est la distribution des contraintes normales au droit des sections?



(a) Cross section.



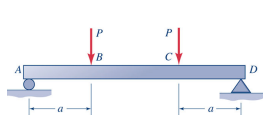
(b) Profile view.

Flexion pure

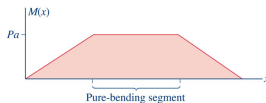
Module 6: éléments en flexion pure

i.e. $M(x) \neq 0$, $V(x) \equiv 0$, $F(x) \equiv 0$

Flexion pure, $V(x) \equiv 0$



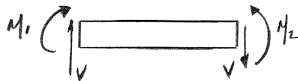
(a) Beam with symmetric loading.



(b) Bending-moment diagram.



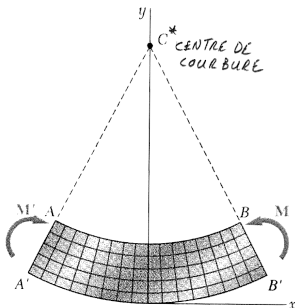
Flexion ordinaire, $V(x) \neq 0$, $M_1 \neq M_2$



Définitions

Courbure: Changement d'angle ($\Delta\theta$) relatif d'un point de la poutre par rapport à un autre

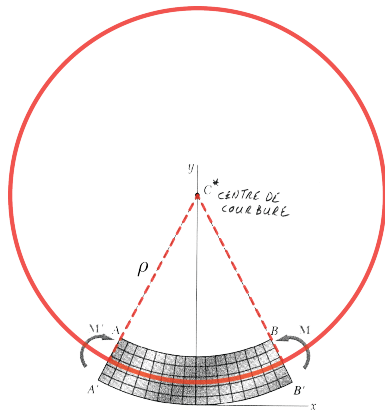
Déflexion: Déplacement relatif (Δy) d'un point de la poutre par rapport à un autre



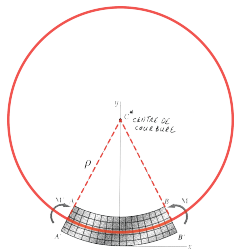
[Beer et al. 2006]

Hypothèses

- ▶ Poutre droite
- ▶ Section constante
- ▶ Section **symétrique**
- ▶ Poutre **supportée latéralement**
- ▶ Matériau **linéaire élastique**
- ▶ **Sections planes** restent planes
- ▶ Déflexion selon un **arc circulaire**
- ▶ Fibres **supérieures comprimées/inférieures tendues**
- ▶ Il existe un **axe neutre (A.N.)**



Courbure (κ) et rayon de courbure (ρ)



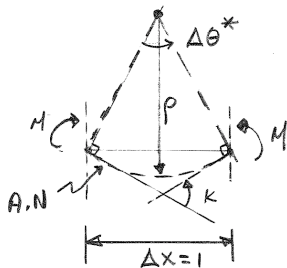
- M : Moment de flexion
- ρ : Rayon de courbure
- $\Delta\theta^*$: Angle entre deux sections séparées par une longueur $\Delta x = 1$
- $\kappa = \Delta\theta^*$: Courbure

Approximation des petits angles

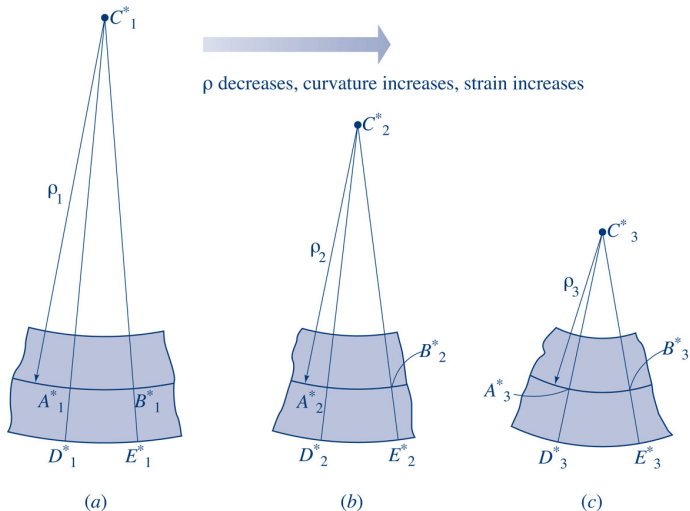
$$\Delta x = \rho \cdot \Delta\theta^* \rightarrow \Delta\theta^* = \frac{\Delta x}{\rho} = \Delta x \cdot \kappa$$

pour $\Delta x = 1 \rightarrow$

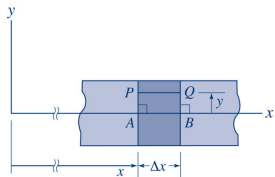
$$\Delta\theta^* = \kappa = \frac{1}{\rho}$$



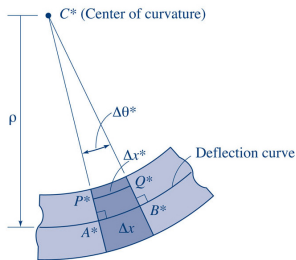
Courbure (κ) et rayon de courbure (ρ) (cont.)



Déformation de flexion – ϵ_x



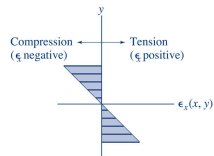
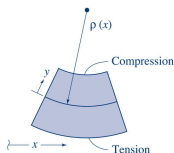
(a) The undeformed beam segment.



(b) The deformed beam segment.

$$\begin{aligned}
 \epsilon_x &\equiv \frac{\overline{P^*Q^*} - \overline{PQ}}{\overline{PQ}} = \frac{\Delta x^* - \Delta x}{\Delta x} \\
 &= \frac{(\rho - y)\Delta\theta^* - \rho\Delta\theta^*}{\rho\Delta\theta^*} \\
 &= \frac{\rho\Delta\theta^* - y\Delta\theta^* - \rho\Delta\theta^*}{\rho\Delta\theta^*} \\
 &= -\frac{y}{\rho} = -\kappa y
 \end{aligned}$$

$\epsilon_x \propto$ **courbure (κ) & distance de l'A.N. (y)**

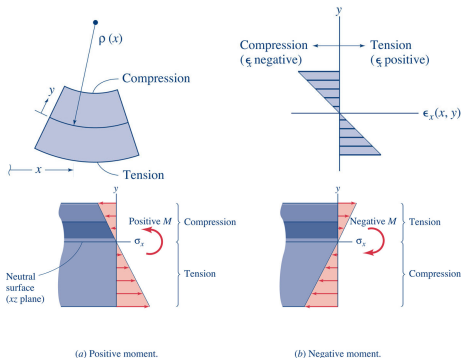


Contraintes de flexion – σ_x

Selon la loi de Hooke

$$\sigma_x = E\epsilon_x = -\frac{Ey}{\rho} = -E\kappa y$$

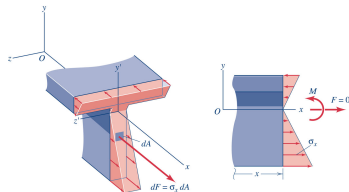
$$\epsilon_x, \sigma_x \propto \kappa, y$$



Pour $M > 0$

- $y > 0, \sigma_x < 0$ (compression)
- $y < 0, \sigma_x > 0$ (traction)
- $y = 0, \sigma_x = 0$ (A.N.)

Équilibre des sections – moments



$$\sum M_z|_{c.g.} = 0 = \underbrace{\int_A \underbrace{y}_{\text{levier}} \cdot \underbrace{\sigma_x \cdot dA}_{\text{force}}}_{\text{moment}} + M$$

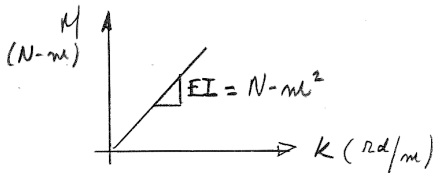
$$M = - \int_A y \cdot \sigma_x \cdot dA$$

$$M = - \int_A -y \frac{Ey}{\rho} dA = \frac{E}{\rho} \underbrace{\int_A y^2 dA}_{\text{inertie, } I_z} = \frac{EI}{\rho} = \kappa EI$$

Relation moment – courbure

- M : Moment de flexion
- EI : Rigidité flexionnelle de la section
- κ : Courbure ($\kappa = \frac{1}{\rho}$)
- ρ : Rayon de courbure

$$M = \frac{EI}{\rho} = \kappa EI \rightarrow \boxed{\kappa = \frac{M}{EI}}$$

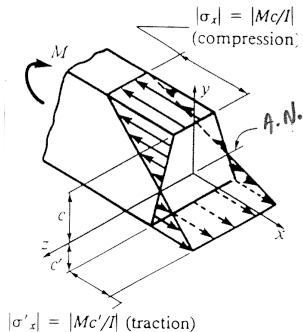


Relation moment – contraintes

$$M = \frac{EI}{\rho} \rightarrow \frac{M}{I} = \frac{E}{\rho}$$

$$\sigma_x = -\frac{Ey}{\rho} \rightarrow -\frac{\sigma_x}{y} = \frac{E}{\rho}$$

$$\frac{M}{I} = -\frac{\sigma_x}{y} \rightarrow \boxed{\sigma_x = -\frac{My}{I}}$$



$$\text{Si } c = |y|_{\max} \rightarrow |\sigma_x|_{\max} = \frac{Mc}{I} = \underbrace{\frac{M}{I/c}}_{\text{Module de section}} = \frac{M}{S}$$

[Bazergui, 2002]

Propriétés des sections communes

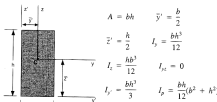
A = area
 $\bar{y}$, $\bar{z}$ = coordinates of centroid C
 I_y , I_z = moments of inertia with respect to the y and z axes, respectively

$I_{y'}$ = moment of inertia with respect to the y' axis
 $I_p = I_y + I_z$ = polar moment of inertia about the centroid C
 I_{yz} = product of inertia with respect to the y and z axes

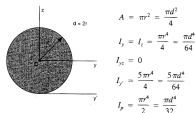
*Basic formulas for area properties may be found in Appendix C.

†(Note: The y and z axes are centroidal axes.)

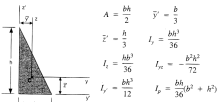
Rectangle



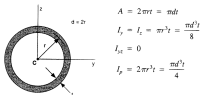
Circle



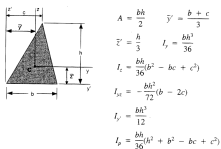
Right Triangle



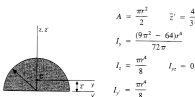
Circular Ring (t/r small)



Triangle

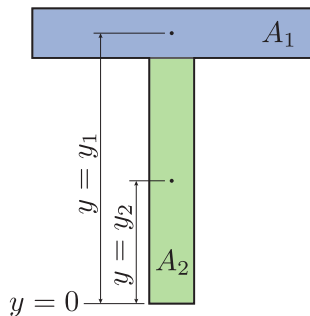


Semicircle



(C.A.H. 2000)

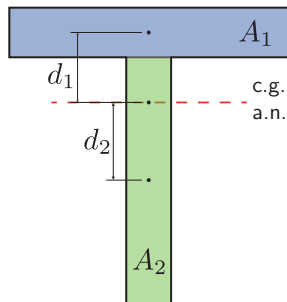
Position de l'axe neutre/centre de gravité – $\bar{y}$



1. Définir un système de référence
2. Subdiviser l'aire de la section A en éléments simples d'aire A_i et ayant un C.G. y_i
3. Calculer le C.G.

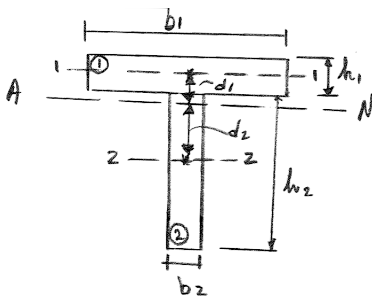
$$\bar{y} = \frac{\sum A_i y_i}{A}$$

L'axe neutre d'une section coïncide avec le centre de gravité

Inertie d'une section (deuxième moment d'aire) – I_z 

1. Subdiviser l'aire de la section A en éléments simples d'aire A_i et ayant un C.G. y_i
2. Calculer le C.G. & A.N.
3. Calculer l'inertie individuelle I_i des éléments
4. Calculer l'inertie de la section (d_i : la distance entre le C.G. de la section et celui de l'élément i)

$$I_z = \sum [I_i + A_i d_i^2]$$

Exemple: Inertie d'une section – I_z 

$$I_1 = \frac{b_1 h_1^3}{12}$$

$$I_2 = \frac{b_2 h_2^3}{12}$$

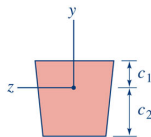
$$I_z = [I_1 + A_1 d_1^2] + [I_2 + A_2 d_2^2]$$

Module de section – S

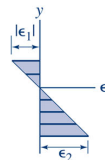
$$\text{Si } c = |y|_{\max} \rightarrow |\sigma_x|_{\max} = \frac{Mc}{I} = \frac{M}{\underbrace{I/c}_{\text{Module de section}}} = \frac{M}{S}$$



(a) Pure bending in longitudinal plane of symmetry.



(b) Cross section.



(c) Strain distribution.

Sections non-symétriques

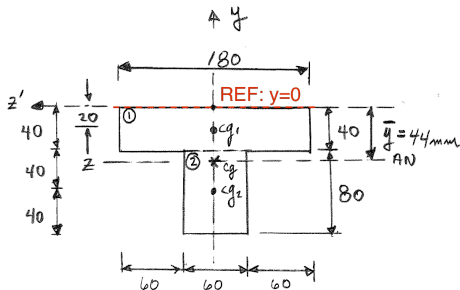
$$S_1 = \frac{I_z}{c_1}, \quad S_2 = \frac{I_z}{c_2}$$

Sections symétriques

$$S_1 = S_2 = \frac{I_z}{c}$$

Exemple – Propriétés des sections

Calculer $\bar{y}$, I_z , et S



(1) Centre de gravité:

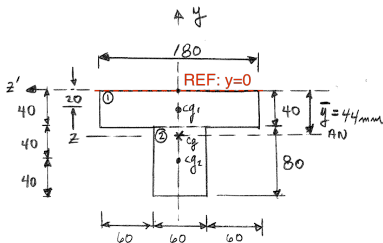
$$\bar{y} = \frac{\sum A_i y_i}{A}$$

(2) Inertie:

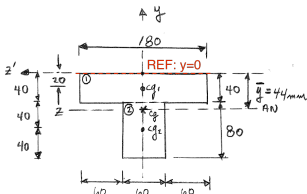
$$I_z = \sum [I_i + A_i d_i^2]$$

(3) Module de section:

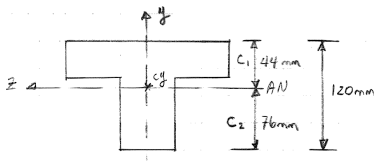
$$S_{\text{sup}} = \frac{I_z}{c_{\text{sup}}}, \quad S_{\text{inf}} = \frac{I_z}{c_{\text{inf}}}$$

Exemple – Propriétés des sections (1) 

$$\begin{aligned}
 \bar{y} &= \frac{\sum A_i y_i}{A} \\
 &= \frac{\underbrace{40\text{mm}}_{h_1} \cdot \underbrace{180\text{mm}}_{b_1} \cdot \underbrace{-20\text{mm}}_{y_1} + 80\text{mm} \cdot 60\text{mm} \cdot -80\text{mm}}{\underbrace{40\text{mm} \cdot 180\text{mm}}_{A_1} + \underbrace{80\text{mm} \cdot 60\text{mm}}_{A_2}} \\
 &= \underline{\underline{-44\text{ mm}}}
 \end{aligned}$$

Exemple – Propriétés des sections (2) 

$$\begin{aligned}
 I_{z,1} &= \sum [I_1 + A_1 d_1^2] \\
 &= \frac{180 \text{ mm} \cdot (40 \text{ mm})^3}{12} + (180 \text{ mm} \cdot 40 \text{ mm}) \cdot (44 \text{ mm} - 20 \text{ mm})^2 \\
 &= \underline{5.11 \times 10^6 \text{ mm}^4} \\
 I_{z,2} &= \underline{8.78 \times 10^6 \text{ mm}^4} \\
 I_z &= I_{z,1} + I_{z,2} = \underline{13.9 \times 10^6 \text{ mm}^4}
 \end{aligned}$$

Exemple – Propriétés des sections (3) 

$$S_{\text{sup.}} = \frac{I_z}{C_{\text{sup.}}} = \frac{13.9 \times 10^6 \text{ mm}^4}{44 \text{ mm}} = \underline{3.16 \times 10^5 \text{ mm}^3}$$

$$S_{\text{inf.}} = \frac{I_z}{C_{\text{inf.}}} = \frac{13.9 \times 10^6 \text{ mm}^4}{120 \text{ mm} - 44 \text{ mm}} = \underline{1.83 \times 10^5 \text{ mm}^3}$$

$$|\sigma_x|_{\text{max}} = \frac{M_z}{S}$$

Contraintes de flexion dans les poutres

$$\sigma_x = -\frac{M_z y}{I_z}$$

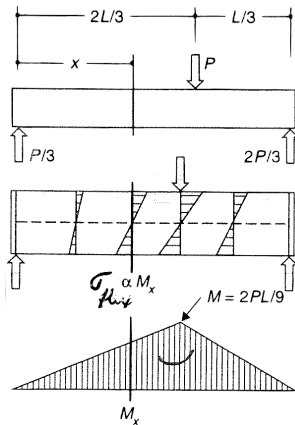
$$|\sigma_x|_{\max} = \frac{M_z c}{I_z} = \frac{M_z}{S}$$

- ▶ σ_x : Contraintes longitudinales de flexion
- ▶ M_z : Moment de flexion auquel doit résister la section
- ▶ I_z : Moment d'inertie de la section par rapport à l'A.N.
- ▶ y : Distance entre l'A.N. et les fibres considérées
- ▶ c : Distance entre l'A.N. et les fibres supérieures/inférieures

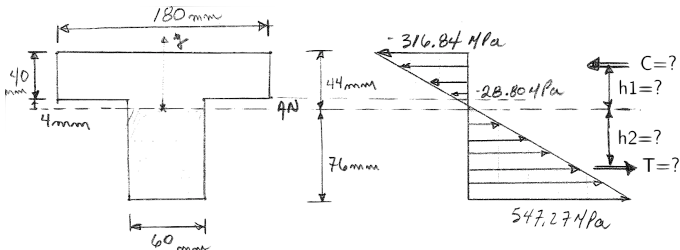
Moments et contraintes de flexion

La magnitude des contraintes est proportionnelle au moment

$$\sigma_x = -\frac{M_z y}{I_z}$$



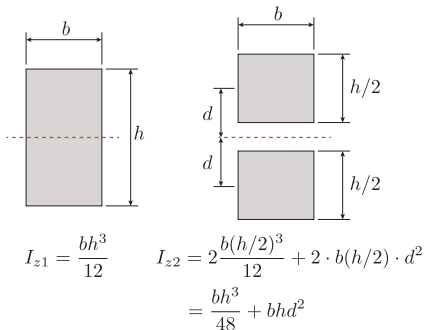
[Schodek, 1998]



Efficacité des sections fléchies

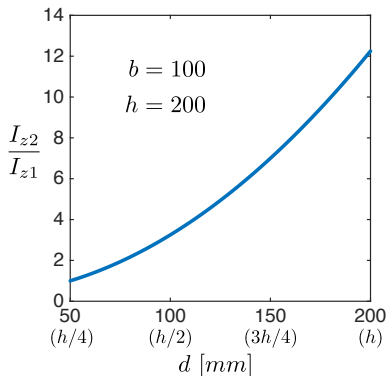
$$\sigma_x = -\frac{M_z y}{I_z}$$

$$I_z = \int_A y^2 dA = \sum [I_i + A_i d_i^2]$$

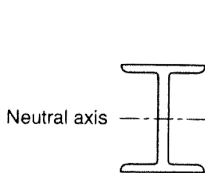


$$I_{z2} > I_{z1}, \text{ pour } d > h/4$$

Quelle forme minimise $|\sigma_x|_{\max}$ et la quantité de matériel?



Efficacité des sections fléchies (cont.)



(a) Steel wide-flange beam.



(b) Plate girder



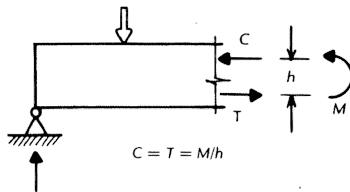
(c) Plywood box beam.

Comparaison de sections communes

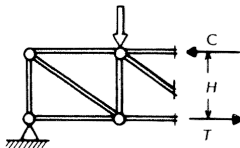
	Aire		Second moment de la section		Module de la section	
	A (mm ²)		I_z (10 ⁶ mm ⁴)		S_z (10 ³ mm ³)	
1)	3630		0,1		11	
2)	3630		1,1		36,5	
3)	3500		1,59		31,1	
4)	3630		1,78		34,8	
5)	3620		5,76		113	
6)	3620		8,35		132	
7)	3330		7,43		105	
8)	3630		1,05		30,9	

[Bazergui, 2002]

Bras de levier interne



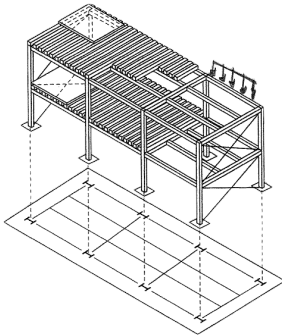
Small h (structural depth), low stiffness



Large structural depth, high stiffness, less material

[Shtakel, 1998]

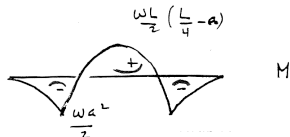
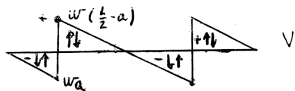
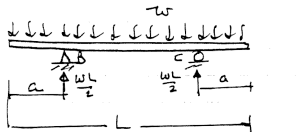
Pourquoi utilise-t-on des poutres pleines?



[Shodek (1998), Google images]

Efforts internes et dimensionnement

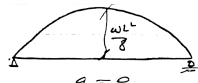
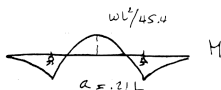
Déterminer a afin de minimiser σ



$$\frac{wa^2}{2} = \frac{wL}{2} \left(\frac{L}{4} - a \right)$$

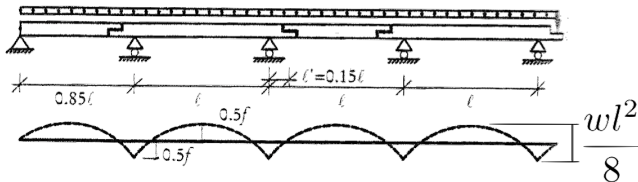
$$a = \frac{L}{2}(\sqrt{2} - 1) \approx 0.21L$$

$$M = \frac{wL}{2} \left(\frac{L}{4} - 0.21L \right) \approx \frac{wL^2}{45.4}$$



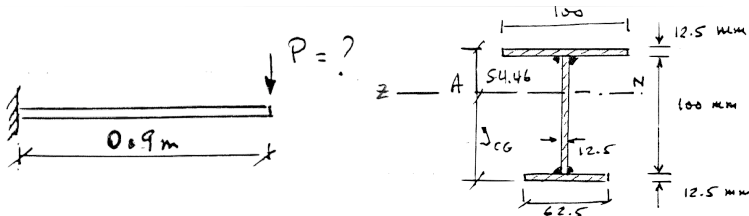
[Bazergui, 2000]

Poutres Gerber



- ▶ Les poutres Gerber sont **isostatiques**
- ▶ Rotules **positionnées afin d'équilibrer** les $M+$ et $M-$ ($\approx 0.15\ell$)
- ▶ La différence entre $M+$ et $M-$ est **égale à** $\frac{wl^2}{8}$

[Muttoni, 2004]

Exemple – Charge admissible Calculer P_{adm} si $|\sigma_x|_{max} \leq 140 \text{ MPa}$ 

 $\rightarrow \bar{y} = c_{inf.} = \underline{70.5 \text{ mm}}, I_z = \underline{72.8 \times 10^5 \text{ mm}^4}$

$$M \leq \frac{\sigma \cdot I_z}{c_{inf.}}$$

$$0.9 \text{ m} \cdot P \leq \frac{140 \text{ MPa} \cdot 72.8 \times 10^5 \text{ mm}^4}{70.5 \text{ mm}}$$

$$\rightarrow P \leq 16\,100 \text{ N} = \underline{16.1 \text{ kN}}$$

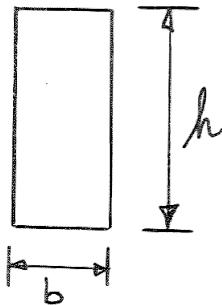
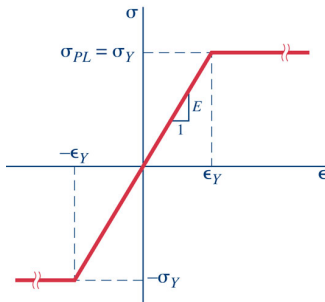
Choix d'une section transversale

Pour **contrainte admissible** $\sigma_{x,adm}$ et un **moment appliqué** M_{max}

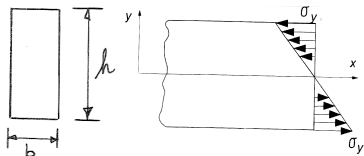
$$|\sigma_x|_{\max} = \frac{M_z}{S} \rightarrow S_{\text{conception}} \geq \frac{|M_{\max}|}{\sigma_{x,adm}}$$

Comportement inélastique des poutres en acier

Cas particulier : poutre à section rectangulaire



Comportement élastique: $\sigma_x \leq \sigma_Y$

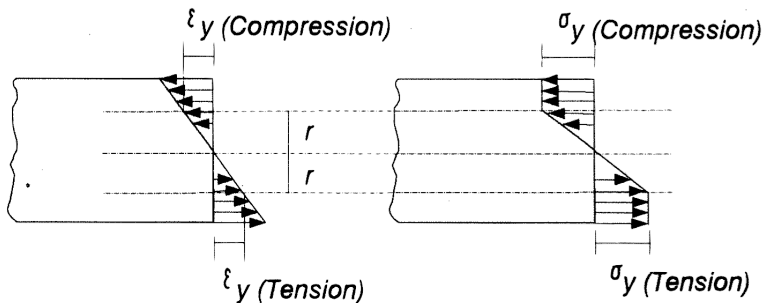


$$\sigma_{x,\max} = \sigma_Y = \frac{M_y c}{I_z} = \frac{M_y \cdot h/2}{b \cdot h^3/12}$$

Moment élastique:

$$M_y = \frac{bh^2}{6} \sigma_Y = \sigma_Y S$$

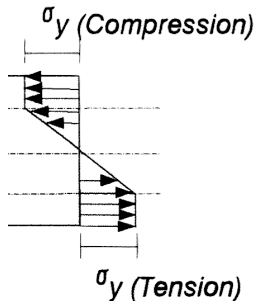
Domaine élastoplastique: $\sigma_x > \sigma_Y$ 



$$\sigma_x = \begin{cases} \frac{y}{r}\sigma_Y, & \text{si } 0 \leq y < r \\ \sigma_Y, & \text{si } r \leq y < h/2 \end{cases}$$

Domaine élastoplastique: $\sigma_x > \sigma_Y$ (cont.)

Moment
élastoplastique:



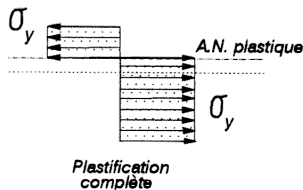
$$\begin{aligned}
 M &= b \int_{-h/2}^{h/2} \sigma_x y dy = 2b \int_0^{h/2} \sigma_x y dy \\
 &= 2b \left[\int_0^r \frac{\sigma_Y}{r} y^2 dy + \int_r^{h/2} \sigma_Y y dy \right] \\
 &= 2b \left[\frac{\sigma_Y}{r} |y^3/3|_0^r + \sigma_Y |y^2/2|_r^{h/2} \right] \\
 &= 2b\sigma_Y \left[\frac{r^2}{3} + \frac{h^2}{8} - \frac{r^2}{2} \right] \\
 &= \sigma_Y \frac{bh^2}{4} \left[1 - \frac{4}{3} \left(\frac{r}{h} \right)^2 \right] \\
 &= \frac{3}{2} M_y \left[1 - \frac{4}{3} \left(\frac{r}{h} \right)^2 \right]
 \end{aligned}$$

Domaine plastique: $\sigma_x > \sigma_Y$, $r = 0$ 

Moment plastique (section rectangulaire):

$$M_p = \sigma_Y \frac{bh^2}{4} \left[1 - \frac{4}{3} \left(\frac{r}{h} \right)^2 \right]$$

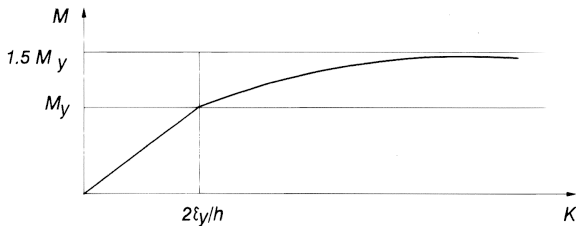
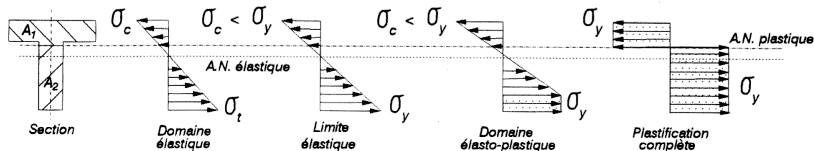
$$= \sigma_Y \frac{bh^2}{4} = \sigma_Y Z = \frac{3}{2} M_y$$



Moment plastique: Moment ultime qu'une section peut supporter.

Z : **module plastique** de la section

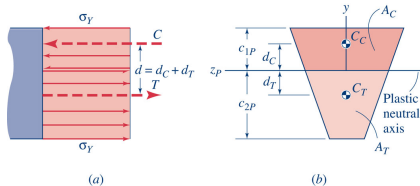
Relation moment–courbure



Coefficient de forme:

$$f = \frac{M_p}{M_y} = \frac{Z}{S}$$

Domaine plastique: $\sigma_x > \sigma_Y$, $r = 0$, cas général



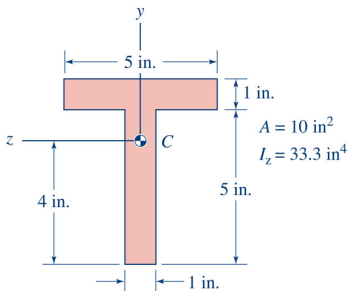
Comme $\sigma_x(y, z) = \sigma_Y$, les aires en traction et en compression doivent être égales ($\sum F_x = 0$)

$$A_C = A_T = \frac{1}{2}A$$

Le **moment plastique** est alors

$$M_p = \frac{\sigma_Y A}{2} \cdot d_C + \frac{\sigma_Y A}{2} \cdot d_T = \frac{\sigma_Y A}{2} (d_C + d_T) = \sigma_Y Z$$

Exemple – flexion plastique 🧩



$$\sigma_Y = 40 \text{ ksi}$$

Calculer

- ▶ Moment élastique, $M_Y = \frac{\sigma_Y I_z}{c}$
- ▶ A.N. plastique, $\bar{y}_p$
- ▶ Moment plastique, M_p
- ▶ Facteur de forme, $f = \frac{M_p}{M_Y}$

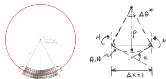
Résumé – Module #6a

But : Calculer les contraintes de flexion

Flexion pure : $M(x) \neq 0$, $V(x) \equiv 0$, $F(x) \equiv 0$



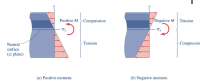
Courbure : $\kappa = 1/\rho$



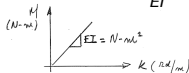
Déformations et contraintes : $\epsilon_x, \sigma_x \propto \kappa, y$

$$\epsilon_x = \frac{-y}{\rho} = -\kappa y$$

$$\sigma_x = \epsilon_x E = \frac{-Ey}{\rho} = -E\kappa y = \boxed{-\frac{My}{I_z}}$$



Relation moment-courbure : $\kappa = \frac{M}{EI}$



Axe neutre : $\bar{y} = \frac{\sum A_i y_i}{\sum A_i} \neq \bar{y}_p$



Inertie : $I_z = \sum [I_i + A_i d_i^2]$, $I_{z, \text{rect.}} = \frac{bh^3}{12}$



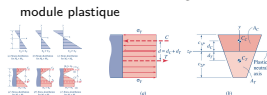
Module de section : $S = \frac{I_z}{c}$, $|\sigma_{x, \text{max}}| = \frac{M_z}{S}$



Conception : $S_{\text{adm.}} \geq \frac{|M_{\text{max}}|}{\sigma_{x, \text{adm.}}}$

Moment plastique :

$M_p = \sigma_Y \cdot \underbrace{Z}_{\text{module plastique}}, \quad f = \frac{Z}{S}$



Organisation de la matière

1 Statique	- Équilibre des forces et moments - Diagrammes de corps libres  - 5 Diagramme des efforts, $N(x)$, $V(x)$, $M(x)$
2 Matériau	- Contraintes & déformations - Loi de Hooke, Poisson & St-Venant 
Chargements	- 3 Efforts axiaux  - 4 Torsion  - 6a Flexion  - 6b Cisaillement  - 7 Déflexion  - 9 Pression & chargements combinés 
États limites	- 7 Déflexion  - 8 Contraintes 2D-3D  - 10 Lois constitutives & critères de rupture 