

Module #2

Contraintes & Déformations

(CIV1150 - Résistance des matériaux)

Enseignant: James-A. Goulet

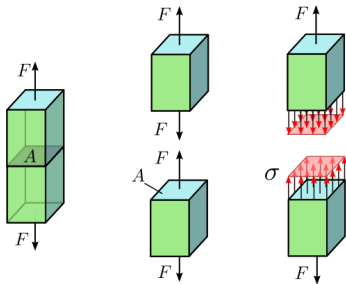
Département des génies civil, géologique et des mines
Polytechnique Montréal



Sections 2.1-2.13 – R. Craig (2011) P. Léger (2006)
Mechanics of Materials, 3rd Edition *Notes de cours: Chapitre 2 – Déformations*
John Wiley & Sons. Polytechnique Montréal.

Contrainte moyenne normale, σ

$$\sigma = \frac{F}{A} = \frac{\text{force normale (i.e. perpendiculaire) à la surface}}{\text{surface sur laquelle elle agit}}$$

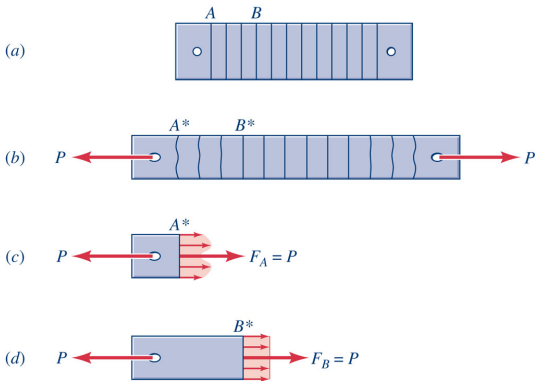


Unités:

- ▶ Pascal: $Pa = N/m^2$
- ▶ **Mégapascal: $MPa = N/mm^2 = 10^6 N/m^2$**
- ▶ $6895 Pa = 1 lb/in^2 = 1 psi$

Signes: + (tension), - (compression)

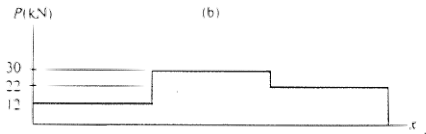
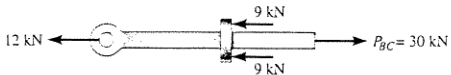
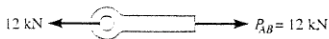
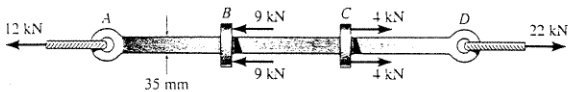
Contrainte moyenne normale



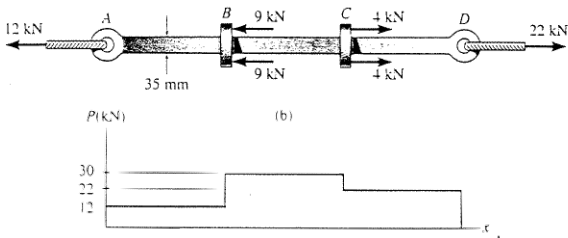
Contrainte moyenne normale: égale pour toutes les sections

Contraintes normales - exemple

Soit une barre ayant une section de 35mm par 10mm tel que



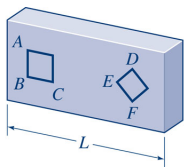
Contraintes normales - exemple (cont.)



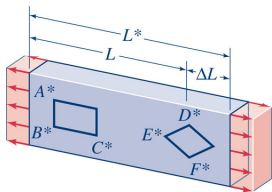
La charge maximale se trouve sur la section BC; $F = 30 \text{ kN}$

$$\sigma = \frac{F}{A} = \frac{30 \times 10^3 \text{ N}}{35 \times 10 \text{ mm}^2} = 85.7 \text{ MPa} \quad \left(\begin{array}{c} \oplus \\ \ominus \end{array} \right)$$

Déformations axiales - ϵ



(a) The undeformed bar.



(b) The deformed bar.

Définition:

$$\begin{aligned}\epsilon &= \frac{\text{longueur finale} - \text{longueur initiale}}{\text{longueur initiale}} \\ &= \frac{L^* - L}{L} \\ &= \frac{\Delta L}{L}\end{aligned}$$

Unités: m/m, mm/mm, in/in,
(microdéformation $\equiv \mu\epsilon = 10^{-6}\epsilon$)

Signe: + Allongement - rétrécissement

Déformations thermiques - 1D

Définition:

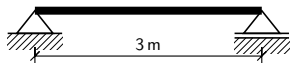
$$\epsilon_T = \alpha \cdot (T_{\text{finale}} - T_{\text{initiale}}) = \alpha \Delta T$$

Unités (α): $mm/mm/^\circ C = /^\circ C$

Coefficient d'expansion thermique (réf. Annexe F-4):

- ▶ Acier: $\alpha = 12 \times 10^{-6}/^\circ C$
- ▶ Aluminium: $\alpha = 21 \text{ à } 24 \times 10^{-6}/^\circ C$
- ▶ Béton: $\alpha = 10 \times 10^{-6}/^\circ C$

Déformations thermiques 1D – exemple cas iso.

Calculer ϵ_T , δ_T et σ_T 

- ▶ $T_{\text{initiale}} = 20^\circ\text{C}$
- ▶ $T_{\text{finale}} = 10^\circ\text{C}$
- ▶ $\alpha = 12 \times 10^{-6}/^\circ\text{C}$

Déformation:

$$\begin{aligned}\epsilon_T &= \alpha \cdot (T_{\text{finale}} - T_{\text{initiale}}) = \alpha \Delta T \\ &= 12 \times 10^{-6}/^\circ\text{C} \cdot -10^\circ\text{C} \\ &= -12 \times 10^{-5}\end{aligned}$$

Déplacement:

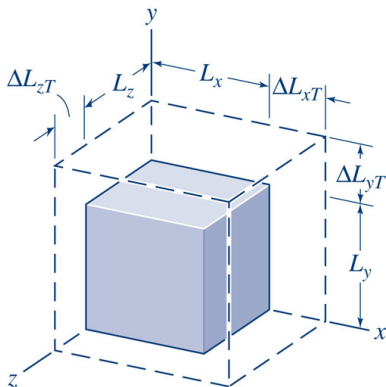
$$\begin{aligned}\delta_T &= \epsilon_T L \\ &= -12 \times 10^{-5} \cdot 3000 \text{ mm} \\ &= -0.36 \text{ mm}\end{aligned}$$

Contrainte:

$$\sigma_T = 0$$

? → **Module 3 – structures hyperstatiques**

Déformations thermiques - 3D 🎥



$$\Delta L_{xT} = (\alpha \Delta T) L_x$$

$$\Delta L_{yT} = (\alpha \Delta T) L_y$$

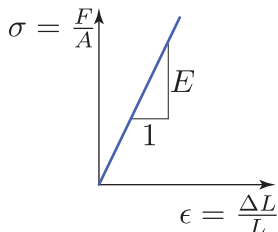
$$\Delta L_{zT} = (\alpha \Delta T) L_z$$

Diagrammes contraintes-déformations, $\sigma = f(\epsilon)$

Contraintes: Effet des **forces internes** sur le matériau

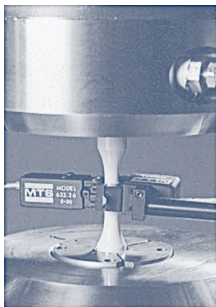
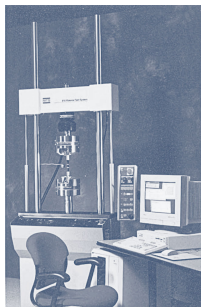
Déformations: Effet des **déplacements internes** sur le matériau

La relation entre les contraintes et les déformations est décrite par le **diagramme contrainte-déformation d'un matériau**



Essais en laboratoires $\sigma - \epsilon$

Le **diagramme contrainte-déformation** est obtenu en laboratoire 

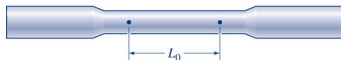


L_0 : Longueur initiale
 A_0 : Aire initiale

$$\Delta L = L^* - L_0$$

$$\sigma = \frac{F}{A_0}$$

$$\epsilon = \frac{\Delta L}{L_0}$$

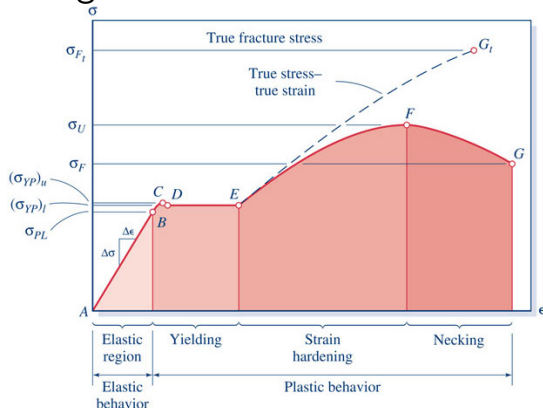


(a) Undeformed specimen.



(b) Deformed specimen.

Diagramme contraintes-déformations – Acier 🎥



σ_{PL} : Limite de proportionnalité
 $(\sigma_{YP})_l$: Limite élastique (σ_y, f_y)
 $(\sigma_{YP})_u$: Limite élastique supérieure

σ_U : Contrainte ultime
 σ_F : Contrainte de rupture
 σ_{F_t} : Contrainte de rupture réelle

Module élastique/de Young

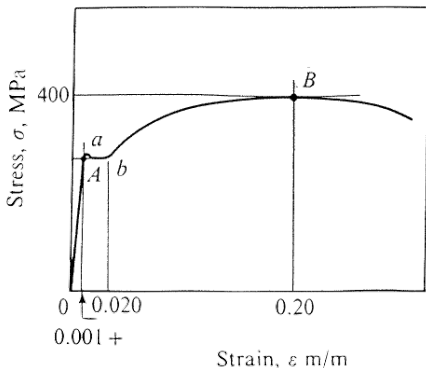
$$E = \frac{\Delta\sigma}{\Delta\epsilon}, \quad \sigma < \sigma_{PL}$$

(E: propriété intrinsèque)

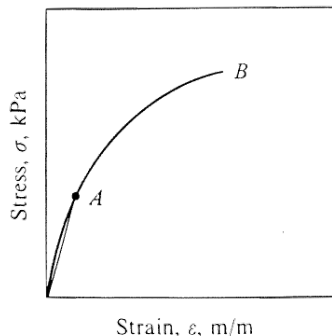
Yielding : plastification
 Hardening : écrouissage
 Necking : amincissement

Diagrammes contraintes-déformations

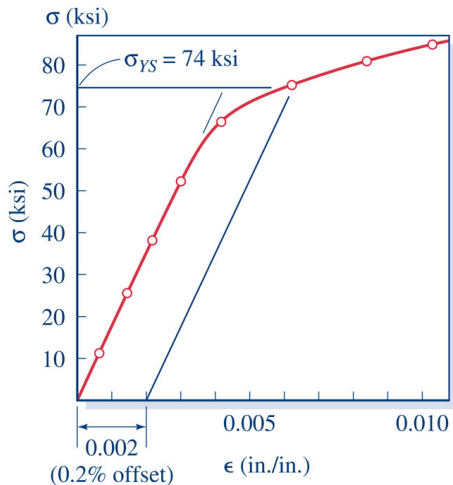
Essai de traction (Acier)



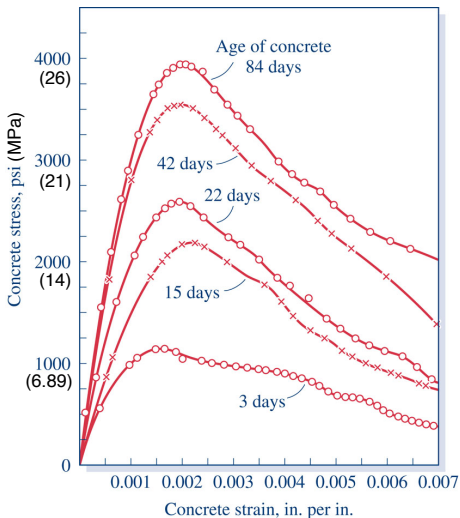
Essai de compression (béton)



Limite élastique - Béton et aluminium

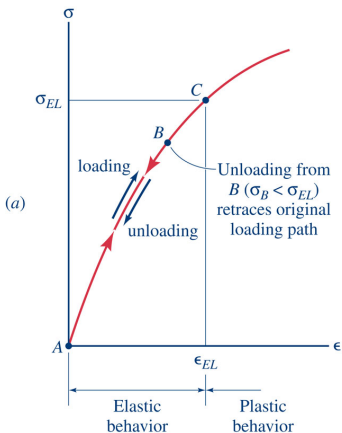


Diagrammes contraintes-déformations – Béton

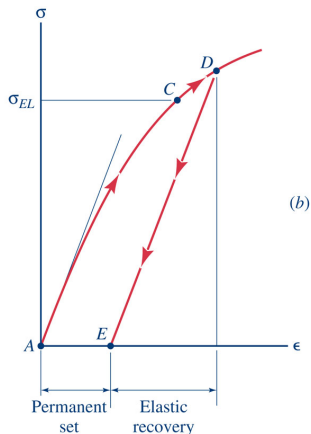


Élasticité

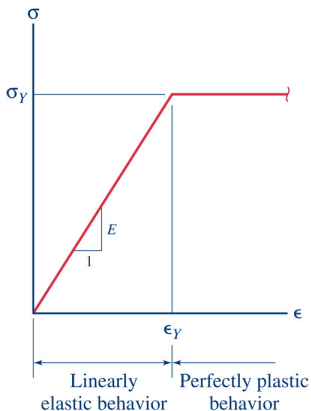
Comportement élastique (non linéaire)



Comportement élastoplastique



Élasticité - Modèle courant



**Comportement élastique,
parfaitement plastique**

Acier courant

- ▶ $E = 200\,000 \text{ MPa} = 200 \text{ GPa}$
- ▶ $\sigma_y = (300, 350) \text{ MPa}$
- ▶ $\epsilon_y \approx 0.0012$

**Même comportement en
traction et en compression**

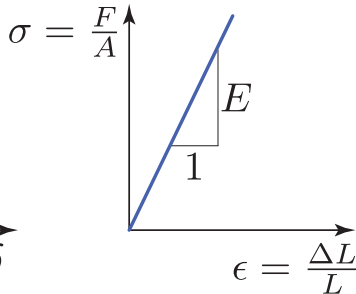
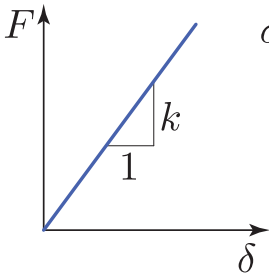
Rigidité & Élasticité

k : Coeff. de rigidité [kN/m]

$$F = k\delta$$

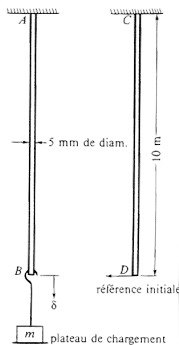
E : Module élastique [MPa]

$$\sigma = \epsilon E$$



$$\sigma = \frac{F}{A} = \epsilon E$$

Rigidité & Élasticité – exemple



$$F = k\delta$$

$$mg = k\delta$$

$$k = \frac{mg}{\delta} = \frac{13.8 \text{ kg} \cdot 9.81 \text{ m/s}^2}{1 \text{ mm}} = 135 \text{ N/mm}$$

$$\sigma = \frac{F}{A} = \frac{40 \times 9.81}{\pi \times 2.5^2} = 20 \frac{\text{N}}{\text{mm}^2} = 20 \text{ MPa}$$

$$\epsilon = \frac{\delta}{L} = \frac{2.9 \text{ mm}}{10000 \text{ mm}} = 2.9 \times 10^{-4}$$

$$\sigma = \epsilon E$$

$$E = \frac{\sigma}{\epsilon} = \frac{20 \text{ MPa}}{2.9 \times 10^{-4}} = 70000 \text{ MPa}$$

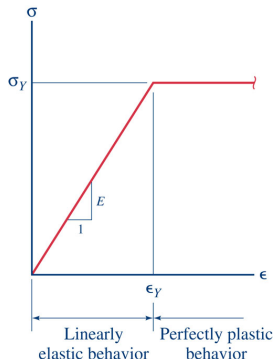
Données

m (kg)	δ_B (mm)
10	0,75
20	1,45
30	2,20
40	2,90

Loi de Hooke

$$\sigma = \epsilon E$$

Matériaux	E[MPa]
Aluminium	70 000
Béton (compression)	30 000
Acier	200 000
Bois (anisotrope)	12 000



La relation entre la contrainte et la déformation est quasi-linéaire pour le domaine élastique des matériaux du génie civil

Coefficient de poisson

Déformation longitudinale

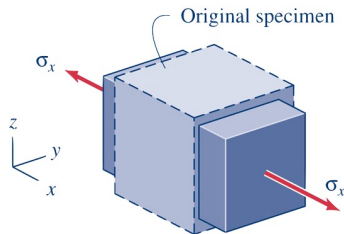
$$\sigma_x = \epsilon_x E \rightarrow \epsilon_x = \frac{\sigma_x}{E}$$

ν : Coefficient de Poisson

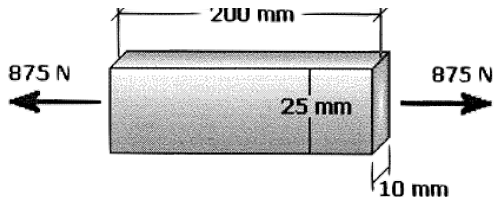
$$\epsilon_{\text{transv.}} = -\nu \epsilon_{\text{long.}}$$

Déformations transversales

$$\epsilon_y = \epsilon_z = -\nu \epsilon_x = -\nu \frac{\sigma_x}{E}$$



Coefficient de poisson - Exemple

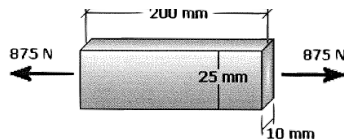


$$\delta_{\text{transv.}} = -0.025 \text{ mm}$$

$$\delta_{\text{long.}} = 0.5 \text{ mm}$$

$$E = ?, \nu = ?$$

Coefficient de poisson - Exemple



$$\sigma = \epsilon E, \nu = -\frac{\epsilon_{\text{transv.}}}{\epsilon_{\text{long.}}}$$

$$\delta_{\text{transv.}} = -0.025 \text{ mm}$$

$$\delta_{\text{long.}} = 0.5 \text{ mm}$$

- ▶ $\sigma = \frac{F}{A} = \frac{875 \text{ N}}{10 \times 25 \text{ mm}^2} = 3.5 \text{ MPa}$
- ▶ $\epsilon_{\text{long.}} = \frac{\Delta L}{L} = \frac{0.5 \text{ mm}}{200 \text{ mm}} = 0.0025 \text{ mm/mm}$
- ▶ $E = \frac{\sigma}{\epsilon} = \frac{3.5 \text{ MPa}}{0.0025 \text{ mm/mm}} = 1400 \text{ MPa}$
- ▶ $\epsilon_{\text{transv.}} = \frac{\Delta \text{ largeur}}{\text{largeur}} = -\frac{0.025 \text{ mm}}{25 \text{ mm}} = -0.0010 \text{ mm}$
- ▶ $\nu = \frac{\epsilon_{\text{transv.}}}{\epsilon_{\text{long.}}} = -\frac{-0.0010}{0.0025} = 0.4$

Coefficient de poisson - Valeurs typiques

< 0



Matériaux artificiels (auxétique)

≈ 0



≈ 0.2



≈ 0.3

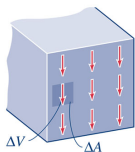


≈ 0.5

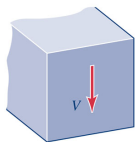


Matériaux communs

Contraintes de cisaillement



(a) The distribution of shear force on a sectioning plane.

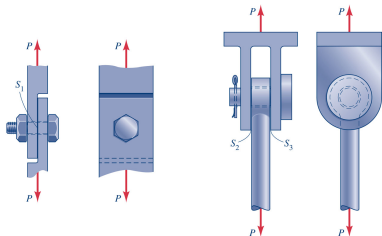


(b) The resultant shear force on the sectioning plane.

- τ : Contrainte de cisaillement
- V : Force de cisaillement
- A_S : Aire effective en cisaillement

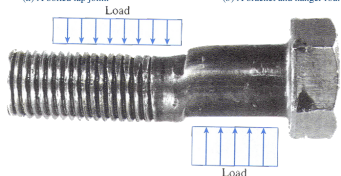
$$\tau = \frac{V}{A_S}$$

Cisaillement pur – Assemblages boulonnés



(a) A bolted lap joint.

(b) A bracket and hanger rod.



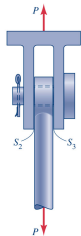
- τ : Contrainte de cisaillement
- V : Force de cisaillement
- A_S : Aire effective en cisaillement

$$\tau = \frac{V}{A_S}$$

Assemblages boulonnés – exemple



(a) A bolted lap joint.



(b) A bracket and hanger rod.

Diamètre des boulons: d

Calculer τ pour les assemblages (a) et (b)?

a) $V = P$

$$A_S = \frac{\pi d^2}{4}$$

$$\tau = \frac{4P}{\pi d^2}$$

b) $V = P/2$

$$A_S = \frac{\pi d^2}{4}$$

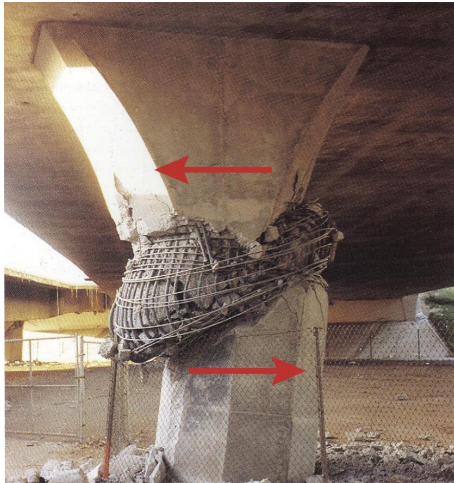
$$\tau = \frac{2P}{\pi d^2}$$

Cisaillement – Poinçonnement, Loma Prieta, 1989

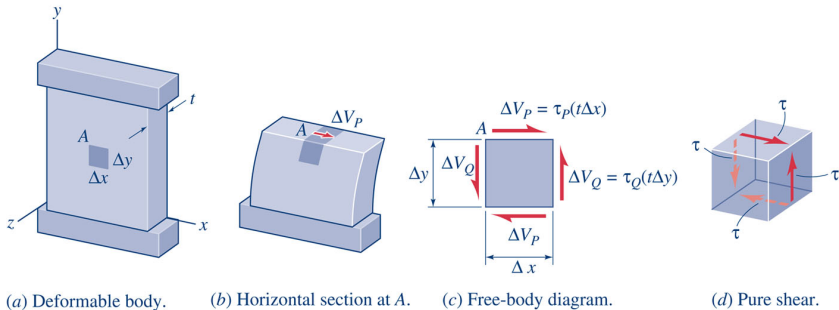


P

Cisaillement – Northridge, 1994



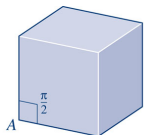
Conditions d'équilibre pour τ



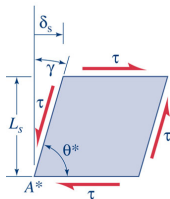
Cisaillement pur

$$\sum F_x = 0, \quad \sum F_y = 0, \quad \sum M_z = 0$$

Déformation en cisaillement γ



(a) Original (undeformed) element.



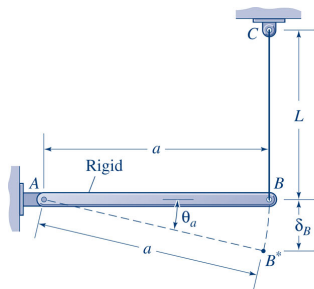
(b) Pure shear deformation.

γ : Changement d'angle dû aux contraintes de cisaillement (effort tranchant)

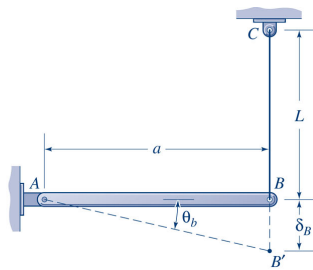
$$\gamma = \frac{\pi}{2} - \theta^* = \underbrace{\tan^{-1} \left(\frac{\delta_s}{L_s} \right)}_{\substack{\text{théorie des petites} \\ \text{déformations}}} \approx \frac{\delta_s}{L_s}$$

⚠ Angles en radians

Hypothèses - petits déplacements & déformations



(a) Actual displacement of B .

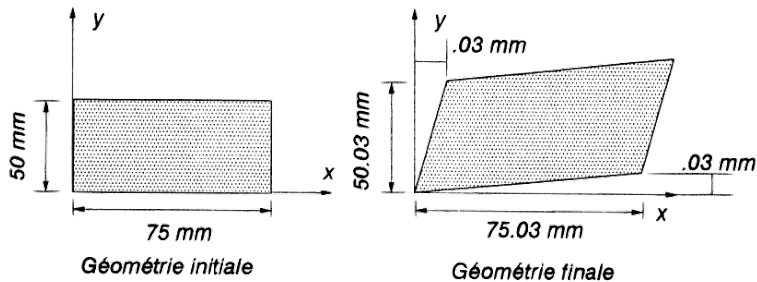


(b) Displacement of B approximated to be vertical.

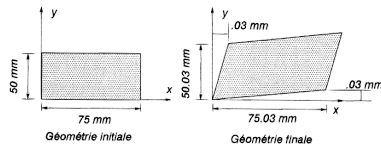
Hypothèse: Les déplacements et déformations sont petits par rapport aux dimensions de la structure

$$\theta_b = \tan^{-1} \left(\frac{\delta_B}{a} \right) \approx \frac{\delta_B}{a} \text{ rad, pour } \theta_b \ll 1 \text{ rad} \quad \text{🎥}$$

Déformation en cisaillement γ – Exemple



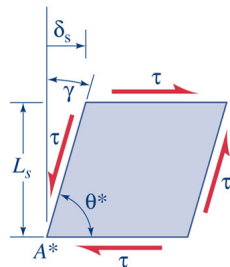
$$\epsilon_x = ?, \quad \epsilon_y = ?, \quad \gamma_{xy} = ?$$

Déformation en cisaillement γ – Exemple

$$\epsilon_x = \frac{75.03 \text{ mm} - 75 \text{ mm}}{75 \text{ mm}} = 0.0004$$

$$\epsilon_y = \frac{50.03 \text{ mm} - 50 \text{ mm}}{50 \text{ mm}} = 0.0006$$

$$\gamma_{xy} = \frac{0.03 \text{ mm}}{75.03 \text{ mm}} + \frac{0.03 \text{ mm}}{50.03 \text{ mm}} = 0.001$$



$$\gamma \approx \frac{\delta_s}{L_s}$$

Loi de Hooke et module de cisaillement

τ : Contrainte de cisaillement

γ : Déformation de cisaillement

G : Module d'élasticité en cisaillement

$$\tau = G\gamma$$

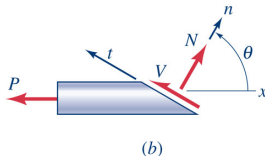
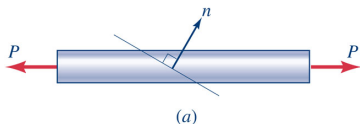
$$G = \frac{E}{2(1 + \nu)}$$

Pour l'acier

- ▶ $E = 200\,000 \text{ MPa}$
- ▶ $\nu = 0.29$
- ▶ $G = 78\,000 \text{ MPa} = 78 \text{ GPa}$

Contraintes – plan incliné

Soit $\sigma_x = \frac{P}{A}$; Quelles sont les contraintes normales (σ_n) et tangentielles (τ_{nt}) selon le système d'axe $n - t$?



$$\sum F_n = 0 : N = P \cos \theta$$

$$\sum F_t = 0 : V = -P \sin \theta$$

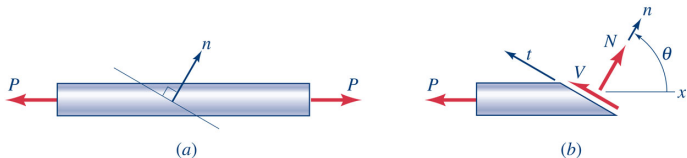
$$A_n = \frac{A}{\cos \theta}$$

$$\sigma_n = \frac{N}{A_n} = \sigma_x \cos^2 \theta$$

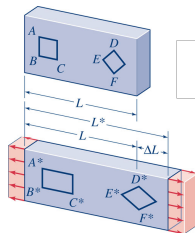
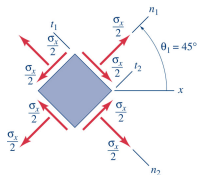
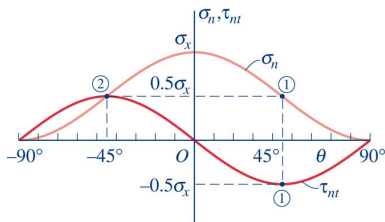
$$\tau_{nt} = \frac{V}{A_n} = -\sigma_x \cos \theta \sin \theta$$

Contraintes sur un plan incliné

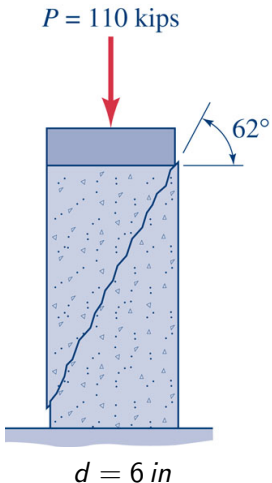
Contraintes – plan incliné (cont.)



$$\sigma_x = \frac{P}{A} \begin{cases} \sigma_n = (\sigma_x/2)(1 + \cos 2\theta) \\ \tau_{nt} = -(\sigma_x/2) \sin 2\theta \end{cases}$$



Contraintes plan incliné – exemple



Calculer σ_n et τ_{nt}

Loi de Hooke (rappel)

Déformation longitudinale

$$\sigma_x = \epsilon_x E \rightarrow \epsilon_x = \frac{\sigma_x}{E}$$

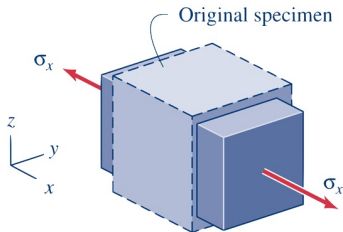
Déformation transversale

$$\epsilon_y = \epsilon_z = -\nu \epsilon_x = -\nu \frac{\sigma_x}{E}$$

Module de cisaillement

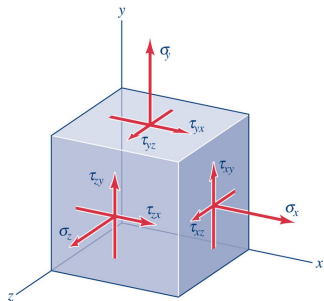
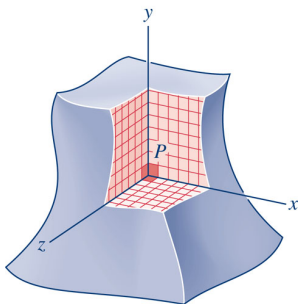
$$G = \frac{E}{2(1 + \nu)}$$

Chargement unidimensionnel

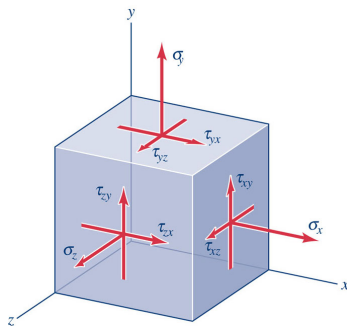


Chargements 2D et 3D?

État des contraintes 3D



Tenseur de contraintes 3D



Tenseur de contraintes

i : Plan normal, j : Direction

$$[\sigma_{ij}] = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix}$$

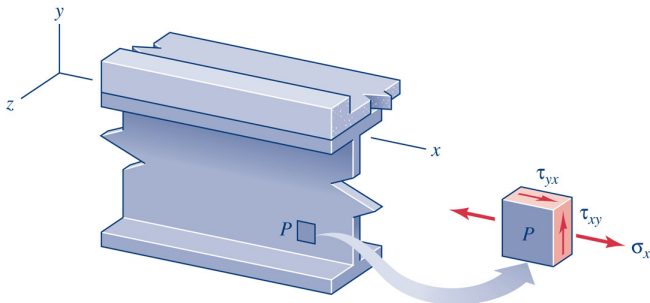
$$[\sigma_{ij}] = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & 0 \\ \tau_{yx} & \sigma_{yy} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Afin de satisfaire les éq. d'équilibre:

$$\tau_{xy} = \tau_{yx}, \tau_{xz} = \tau_{zx}, \tau_{yz} = \tau_{zy}$$

Chargement 2D - contraintes planes

Contraintes planes: $\sigma_z = \tau_{xz} = \tau_{yz} = 0$

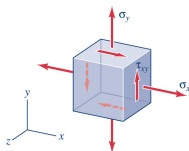


(a) A portion of a bridge girder.

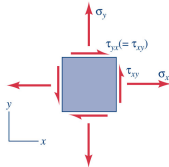
(b) A web element, with stresses shown.

Chargement 2D - contraintes planes

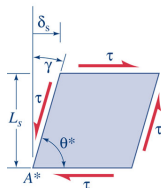
Contraintes planes: $\sigma_z = \tau_{xz} = \tau_{yz} = 0$



(a) Three-dimensional view.



(b) Two-dimensional view.

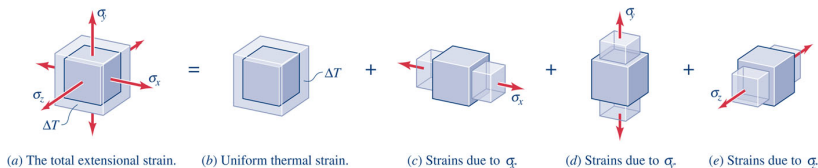


$$\epsilon_x = \frac{\sigma_x}{E} - \frac{\nu\sigma_y}{E} = \frac{1}{E}(\sigma_x - \nu\sigma_y)$$

$$\epsilon_y = \frac{1}{E}(\sigma_y - \nu\sigma_x)$$

$$\gamma_{xy} = \frac{1}{G}\tau_{xy}$$

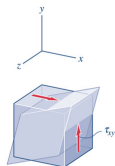
Loi de Hooke généralisée 3D - Déformations axiales



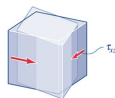
$$\begin{aligned} \epsilon_x &= \frac{1}{E}[\sigma_x - \nu(\sigma_y + \sigma_z)] + \alpha\Delta T \\ \epsilon_y &= \frac{1}{E}[\sigma_y - \nu(\sigma_x + \sigma_z)] + \alpha\Delta T \\ \epsilon_z &= \frac{1}{E}[\sigma_z - \nu(\sigma_x + \sigma_y)] + \alpha\Delta T \end{aligned}$$

Note: Les contraintes de cisaillement n'ont pas d'effet sur les déformations axiales

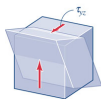
Loi de Hooke généralisée 3D - Déformations de cisaillement



(a) τ_{xy} produces γ_{xy} only.



(b) τ_{xz} produces γ_{xz} only.



(c) τ_{yz} produces γ_{yz} only.

$$\gamma_{xy} = \frac{1}{G} \tau_{xy}$$

$$\gamma_{xz} = \frac{1}{G} \tau_{xz}$$

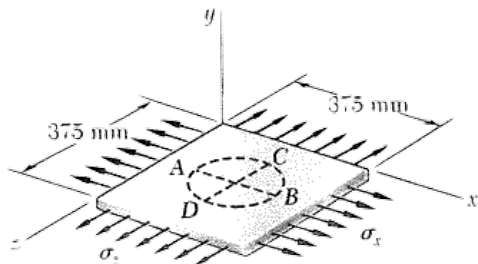
$$\gamma_{yz} = \frac{1}{G} \tau_{yz}$$

- ▶ Pas d'effet de poisson pour les déformations en cisaillement
- ▶ Les contraintes axiales n'ont pas d'effet sur les déformations en cisaillement

Loi de Hooke généralisée 3D - Contraintes

$$\begin{aligned}\sigma_x &= \frac{E}{(1+\nu)(1-2\nu)} [(1+\nu)\epsilon_x + \nu(\epsilon_y + \epsilon_z) - (1+\nu)(\alpha\Delta T)] \\ \sigma_y &= \frac{E}{(1+\nu)(1-2\nu)} [(1+\nu)\epsilon_y + \nu(\epsilon_x + \epsilon_z) - (1+\nu)(\alpha\Delta T)] \\ \sigma_z &= \frac{E}{(1+\nu)(1-2\nu)} [(1+\nu)\epsilon_z + \nu(\epsilon_x + \epsilon_y) - (1+\nu)(\alpha\Delta T)]\end{aligned}$$

$$\tau_{xy} = G\gamma_{xy}, \tau_{xz} = G\gamma_{xz}, \tau_{yz} = G\gamma_{yz}$$

Chargement 2D, contraintes planes – exemple 

- ▶ $t = 25 \text{ mm}$
- ▶ $d = 225 \text{ mm}$
- ▶ $\sigma_x = 82.7 \text{ MPa}$
- ▶ $\sigma_z = 138 \text{ MPa}$
- ▶ $E = 69\,000 \text{ MPa}$
- ▶ $\nu = 1/3$

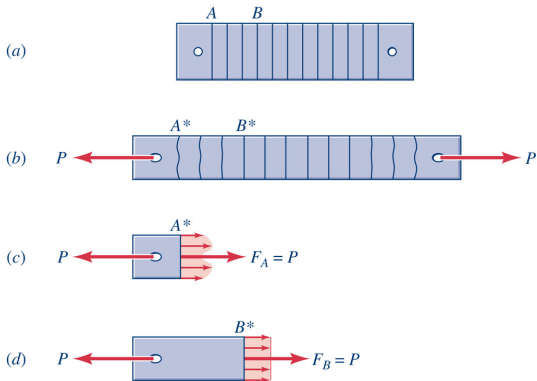
Déterminez les changements

- a) de diamètre AB
- b) de diamètre CD
- c) de l'épaisseur t

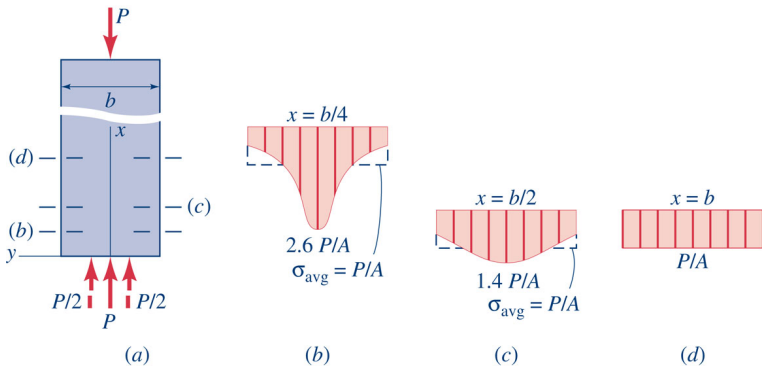
$$\triangle \sigma_y = 0$$

$$\triangle \epsilon_y \neq 0$$

Effet des charges concentrées



Principe de St-Venant



Pour $x \geq b$ la distribution des contraintes est uniforme

$$\sigma = \sigma_{\text{avg}} = \frac{P}{A}, \quad x \geq b$$

Dimensionnement

Défaillance: charges \geq résistance

Facteur de sécurité (défaillance si $FS \leq 1$, typique: 3)

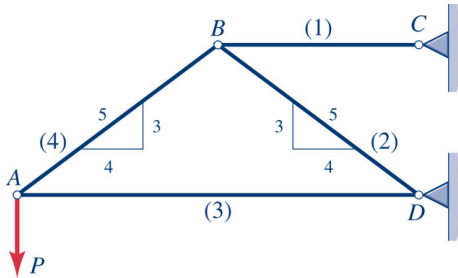
$$FS = \frac{\text{résistance}}{\text{charges}}$$

$$\text{Contrainte admissible} = \frac{\text{limite élastique}}{FS}$$

Note: de nos jours, les structures du génie civil ne sont plus conçues en utilisant le FS

Exemple treillis

Dimensionnement – exemple



$$A = 500 \text{ mm}^2$$

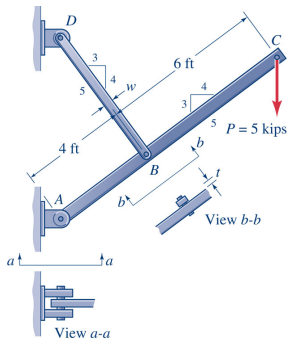
$$\sigma_T = 300 \text{ MPa}$$

$$\sigma_C = 200 \text{ MPa}$$

$$FS = 3$$

Quelle est la charge admissible $P_{adm.}$?

Dimensionnement – exemple



$$w = 2 \text{ in}$$

$$\sigma_y = 36 \text{ ksi}$$

$$\tau_y = 48 \text{ ksi}$$

$$FS = 3$$

Question:

- Dimensionner l'épaisseur du tirant w
- Dimensionner le diamètre de boulons aux connections A et B

Choisir les dimensions au 1/8in près

Résumé – Module #2

Définitions :

$$\left. \begin{array}{l} \epsilon = \Delta L / L \\ \sigma = F / A \end{array} \right\} \text{signes: (+)tension, (-)compression } \left(\begin{array}{c} \oplus \\ \ominus \end{array} \right)$$

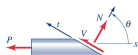
Chargements : force : $\epsilon = \frac{\sigma}{E}$, température : $\epsilon = \alpha \Delta T$

Matériau : Diagramme $\sigma - \epsilon$





Loi de Hooke & coeff. de Poisson : $\sigma_x = \epsilon_x E$, $\epsilon_y = \epsilon_z = \nu \epsilon_x$







Plan incliné : Décomposition des forces





Organisation de la matière

- 1 *Statique*
 - Équilibre des forces et moments
 - Diagrammes de corps libres 
 - 5 Diagramme des efforts, $N(x)$, $V(x)$, $M(x)$

- 2 *Matériau*
 - Contraintes & déformations
 - Loi de Hooke, Poisson & St-Venant 

- Chargements*
 - 3 Efforts axiaux 
 - 4 Torsion 
 - 6a Flexion 
 - 6b Cisaillement 
 - 7 Déflexion 
 - 9 Pression & chargements combinés 

- États limites*
 - 7 Déflexion 
 - 8 Contraintes 2D-3D 
 - 10 Lois constitutives & critères de rupture 