

# Module #10

## Critères de défaillance & concentration de contraintes

(CIV1150 - Résistance des matériaux)

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Polytechnique Montréal



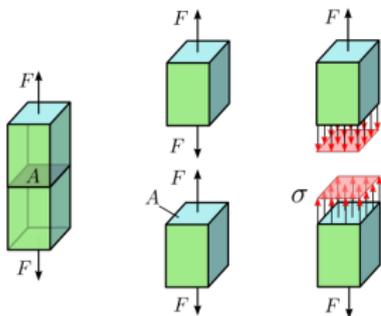
§10.1, 12.1-12.4–R. Craig (2011)  
*Mechanics of Materials*, 3rd Edition  
John Wiley & Sons.

P. Léger (2006)  
*Notes de cours: Chapitre 10, §10.1–10.9*  
*Polytechnique Montréal.*

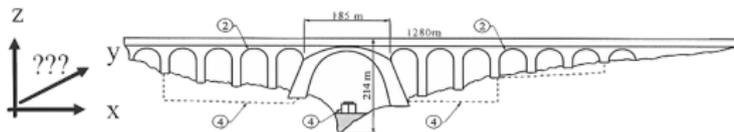
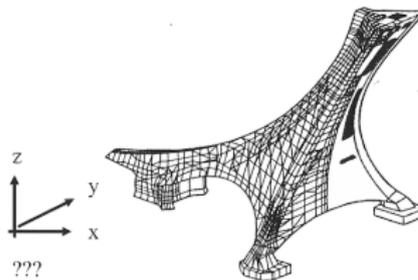
# Mise en contexte – Critères de rupture

Capacité de la barre?

Contraintes principales en 2D-3D

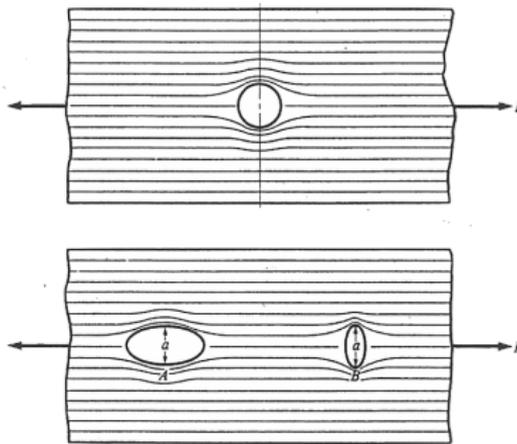


$$F_{adm.} = \sigma_{adm.} \cdot A$$



Quels sont les critères de rupture  
pour les chargements multidirectionnels?

# Mise en contexte – concentration de contraintes



**Quel est l'effet des discontinuités  
sur les concentrations de contraintes?**

[Bazergui, 2002]

## Mise en contexte – Fatigue



**Comment quantifier la résistance à la fatigue?**

[Google Images]

# Mise en contexte – Sécurité des ouvrages

**Quel ouvrage présente le plus grand risque?**

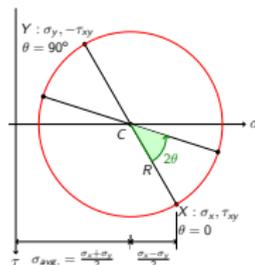


**Risque:** probabilité de défaillance  $\times$  conséquences

[Google Images]

# Objectifs

## Chapitre 8 : Contraintes principales

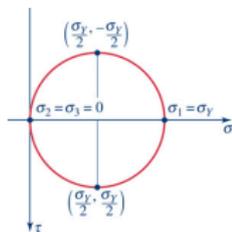
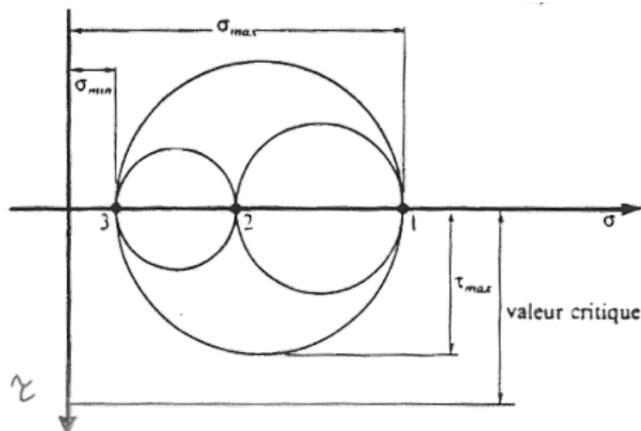


## Chapitre 10 :

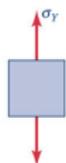
1. Critères de rupture pour chargements multidirectionnels
2. Effets des discontinuités géométriques sur les concentrations de contraintes
3. Résistance à la fatigue
4. Quantification de la sécurité des ouvrages

# Critère de Tresca – contrainte maximale de cisaillement

Écoulement plastique lorsque  $\tau_{max} > \tau_{adm.}$



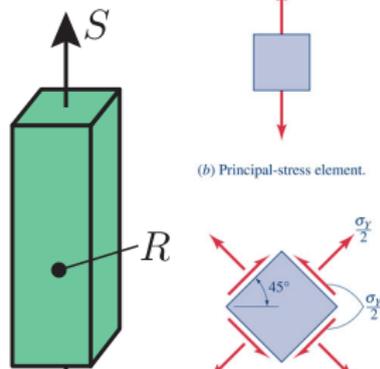
(a) Mohr's circle for  $\sigma_1 = \sigma_y$ .



(b) Principal-stress element.

Pour un essai **uniaxial**:  $\sigma_1 > 0$ ,  $\sigma_2 = \sigma_3 = 0$

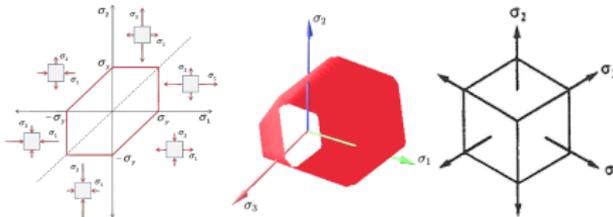
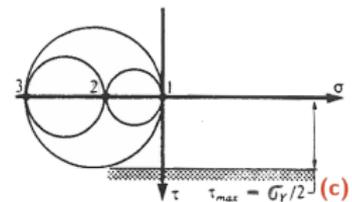
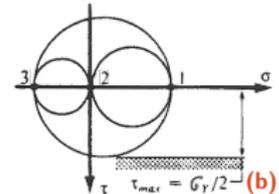
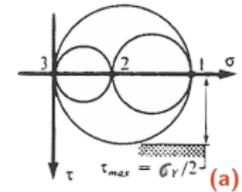
$$\tau_{adm.} = \frac{\sigma_Y}{2}$$



# Critère de Tresca – contrainte maximale de cisaillement

Pour un essai **biaxial** ( $\sigma$  planes):

- a)**  $\sigma_1 \geq 0, \sigma_3 = 0 \rightarrow \tau_{adm.} = \frac{\sigma_Y}{2}$   
plastification lorsque  $\sigma_1 = \sigma_Y$
- b)**  $\sigma_1 \geq 0, \sigma_3 \leq 0 \rightarrow \tau_{adm.} = \frac{\sigma_1 - \sigma_3}{2}$   
plastification lorsque  $\sigma_1 - \sigma_3 > \sigma_Y$
- c)**  $\sigma_1 = 0, \sigma_3 \leq 0 \rightarrow \tau_{adm.} = \frac{\sigma_3}{2}$   
plastification lorsque  $|\sigma_3| = \sigma_Y$



[Bazergui, simulation matériaux]

# Critère de Tresca – exemple

$$\sigma_1 = 208 \text{ MPa}$$

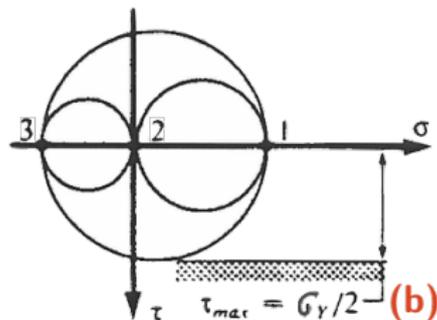
$$\sigma_2 = 200 \text{ MPa}$$

$$\sigma_3 = -108 \text{ MPa}$$

$$\sigma_Y = 270 \text{ MPa}$$

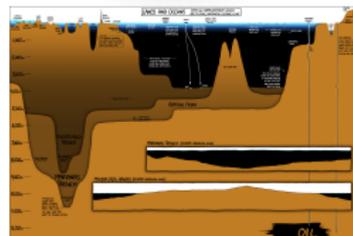
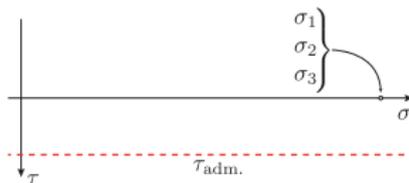
$$\sigma_1 - \sigma_3 = 316 \text{ MPa} > \sigma_Y = 270 \text{ MPa}$$

**⚠ plastification**



# Critère de Tresca – contrainte hydrostatique

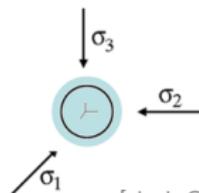
$$\begin{aligned}
 P &= \rho \cdot g \cdot h \\
 &= 1000 \frac{\text{kg}}{\text{m}^3} \times 9.81 \frac{\text{m}}{\text{s}^2} \times 11000 \text{m} \\
 &= 108 \times 10^6 \frac{\text{N}}{\text{m}^2} = 108 \text{ MPa}
 \end{aligned}$$



## Contrainte hydrostatique

Pour  $|\sigma_1 = \sigma_2 = \sigma_3| > \sigma_Y$

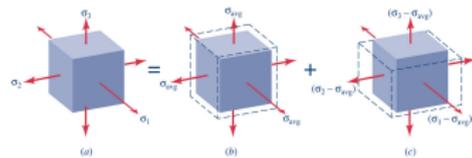
**⚠ pas de plastification**



[xkcd, Google images]

# Critère de Von Mises – Énergie de distorsion maximale

Écoulement plastique lorsque l'**énergie de distorsion** atteint la valeur critique



Énergie de distorsion:  $\bar{u}_d$

$$\bar{u}_d = \frac{1}{12G} [(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_1 - \sigma_3)^2]$$

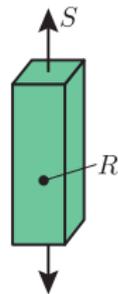
$$\bar{u}_d = \frac{1}{12G} [(\sigma_Y - 0)^2 + (0 - 0)^2 + (\sigma_Y - 0)^2]$$

Chargement uniaxial :

$$\sigma_1 = \sigma_Y, \sigma_2 = \sigma_3 = 0$$

Énergie de distorsion maximale:

$$\bar{u}_d^* = \frac{1}{12G} \cdot 2\sigma_Y^2 = \frac{\sigma_Y^2}{6G}$$



# Critère de Von Mises – Énergie de distorsion maximale

$$\bar{u}_d^* = \bar{u}_d$$

$$\frac{\sigma_Y^2}{6G} = \frac{1}{12G} [(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_1 - \sigma_3)^2]$$

$$\sigma_Y^2 = \frac{1}{2} [(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_1 - \sigma_3)^2]$$

$$\sigma_Y = \sqrt{\frac{1}{2} [(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_1 - \sigma_3)^2]}$$

## Contrainte équivalente de Von Mises

$$\sigma_M = \sqrt{\frac{1}{2} [(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_1 - \sigma_3)^2]}$$

**Plastification:**  $|\sigma_M| \geq \sigma_Y$

## Contrainte équivalente de Von Mises

Contraintes principales –  $\sigma_1, \sigma_2, \sigma_3$

$$\sigma_M = \sqrt{\frac{1}{2} [(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_1 - \sigma_3)^2]}$$

Contraintes –  $\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{yz}, \tau_{xz}$

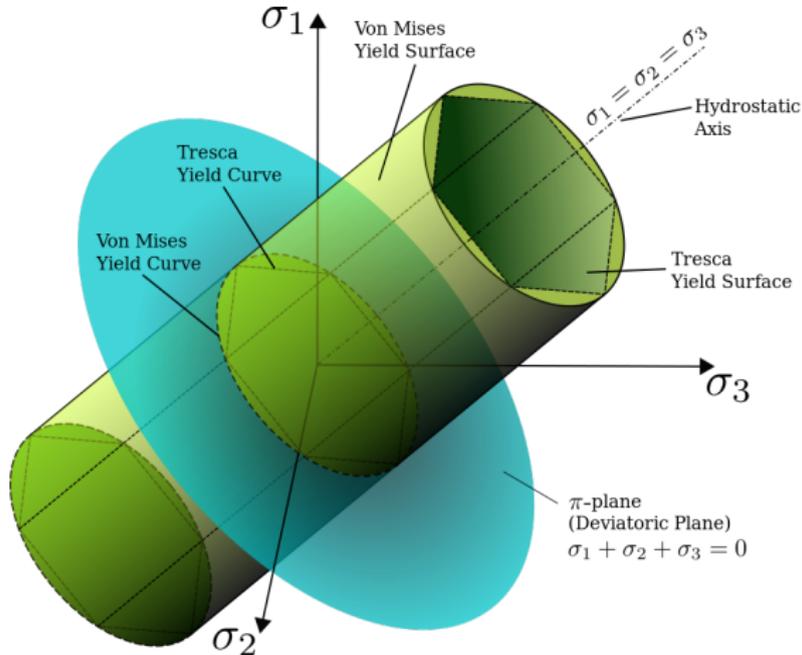
$$\sigma_M = \sqrt{\frac{1}{2} [(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_x - \sigma_z)^2 + 6(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{xz}^2)]}$$

Contraintes planes –  $\sigma_x, \sigma_y, \tau_{xy}$

$$\sigma_M = \sqrt{\sigma_x^2 + \sigma_y^2 - \sigma_x \sigma_y + 3\tau_{xy}^2}$$

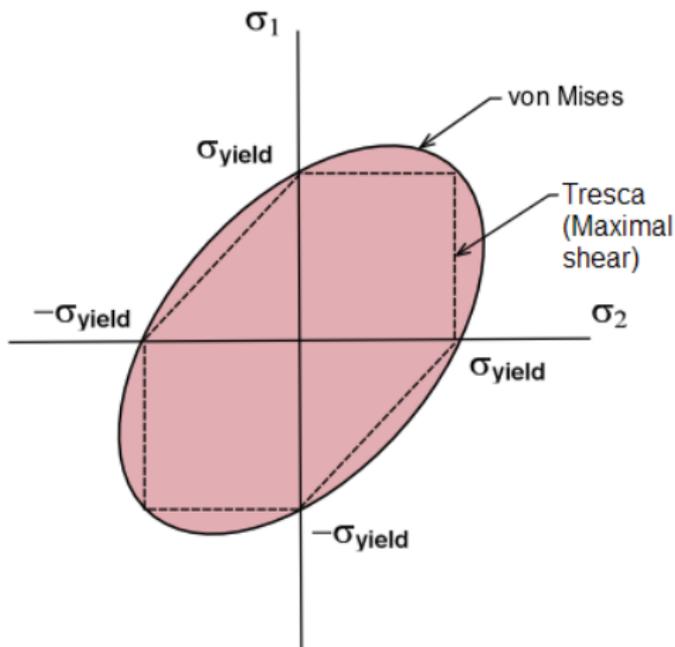
**Plastification:**  $|\sigma_M| \geq \sigma_Y$

# Comparaison 3D Von Mises & Tresca



[wikipedia]

# Comparaison 2D Von Mises & Tresca ( $\sigma$ planes)



[wikipedia]

## Critère de Von Mises – exemple

$$\sigma_1 = 208 \text{ MPa}$$

$$\sigma_2 = 200 \text{ MPa}$$

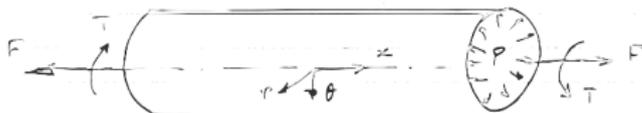
$$\sigma_3 = -108 \text{ MPa}$$

$$\sigma_Y = 270 \text{ MPa}$$

$$\begin{aligned} \sigma_M &= \sqrt{\frac{1}{2} [(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_1 - \sigma_3)^2]} \\ &= \sqrt{\frac{1}{2} [(208 - 200)^2 + (200 + 108)^2 + (208 + 108)^2]} \\ &= 312 \text{ MPa} > \sigma_Y = 270 \text{ MPa} \end{aligned}$$

 **plastification**

# Comparaison Von Mises & Tresca



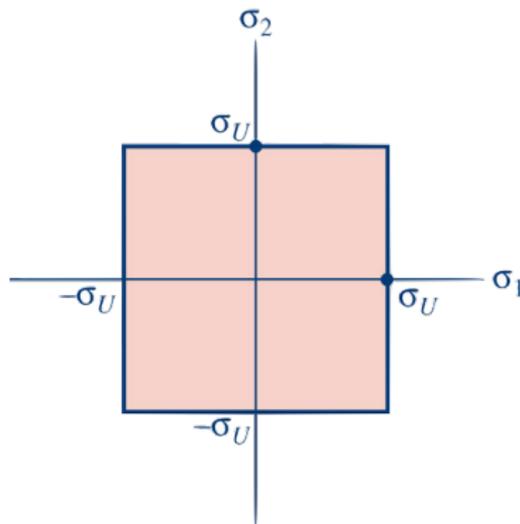
(cylindre ouvert)

- ▶  $p = 0.9 \text{ MPa}$
- ▶  $F = 90 \text{ kN}$
- ▶  $FS = 3.8$
- ▶  $r_{\text{moyen}} = 250 \text{ mm}$
- ▶  $t = 5 \text{ mm}$
- ▶  $\sigma_Y = 190 \text{ MPa}$

Calculer  $T_{adm.}$  pour les critères de **Tresca** et **Von Mises**

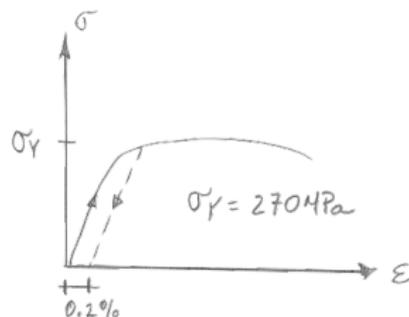
# Critère de Rankine

$$\text{Rupture si } \begin{cases} |\sigma_1| \geq \sigma_U, \text{ ou} \\ |\sigma_2| \geq \sigma_U, \text{ ou} \\ |\sigma_3| \geq \sigma_U \end{cases}$$



## Critère de Rankine – exemple

$$\begin{aligned}\sigma_x &= 200 \text{ MPa} \\ \sigma_y &= -100 \text{ MPa} \\ \sigma_z &= 200 \text{ MPa} \\ \tau_{xy} &= 50 \text{ MPa}\end{aligned}$$

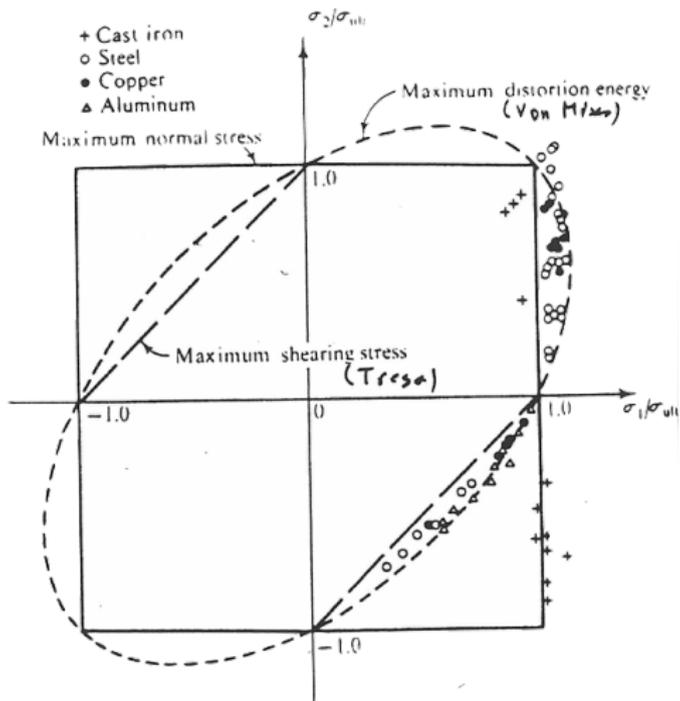


$$\left. \begin{aligned}\sigma_1 &= 208 \text{ MPa} \\ \sigma_2 &= 200 \text{ MPa} \\ \sigma_3 &= -158 \text{ MPa}\end{aligned} \right\} |\sigma_p| < \sigma_Y = 270 \text{ MPa} \triangle$$

Pour un matériau **ductile**, il y aurait rupture...

$$\text{Rankine: } 208 \text{ MPa} + 158 \text{ MPa} = 366 \text{ MPa} > 270 \text{ MPa} \triangle$$

# Essais empiriques ( $\sigma$ plane)



[Popov]

## Critère de Mohr ( $\sigma$ plane)

- ▶ Les matériaux fragiles ne subissent pas de distorsion
- ▶ La structure atomique ne permet pas de glissement
- ▶ Matériaux fragiles: résistance en compression  $>$  résistance en traction (microfissures + concentration de contraintes)

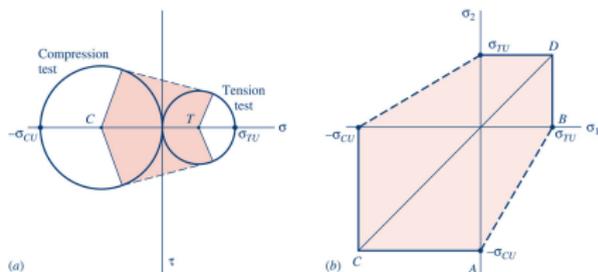
### Critères de rupture:

Pour  $\sigma_1$  et  $\sigma_2$  de même signe :

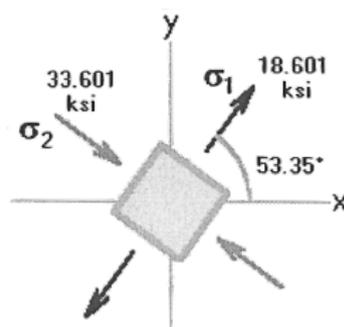
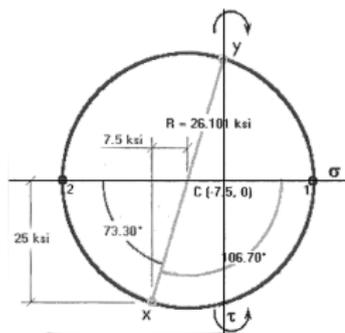
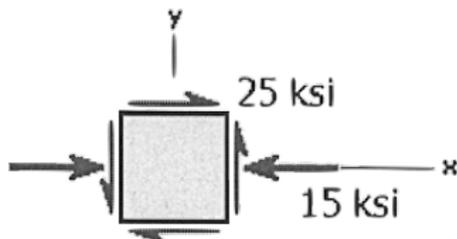
$$\sigma_{\max} = \sigma_{TU} \text{ ou } \sigma_{\min} = -\sigma_{CU}$$

Pour  $\sigma_1 > 0$  et  $\sigma_2 < 0$  :

$$\frac{\sigma_1}{\sigma_{TU}} = \frac{\sigma_2}{\sigma_{CU}} + 1$$



# Comparaison des critères – exemple MDS 12.1

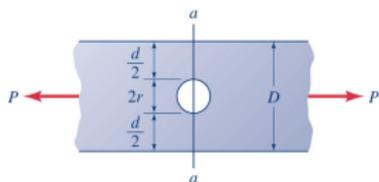


$$\sigma_Y = 50 \text{ ksi}$$

# Concentration de contraintes

$K$ : Facteur de concentration de contraintes

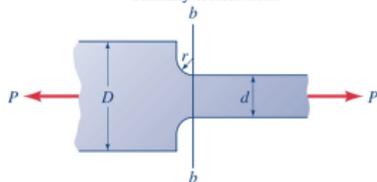
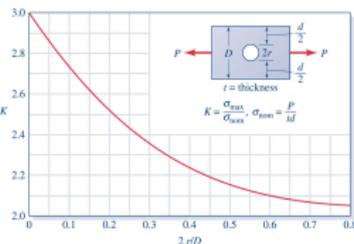
$$K = \frac{\sigma_{\max}}{\sigma_{\text{norm}}} \rightarrow \sigma_{\text{norm}} : \sigma_{\text{avg.}} \text{ à la section réduite}$$



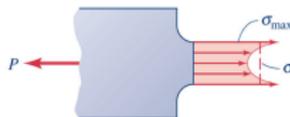
(a) Flat bar with centrally located hole.



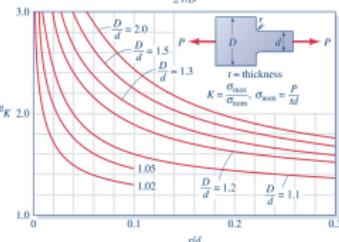
(b) Stress distribution on section  $a-a$ .



(a) Flat bar with reduced section.



(b) Stress distribution on section  $b-b$ .



# Concentration de contraintes – exemple

$$K = 2.45 = \frac{\sigma_{\max}}{\sigma_{\text{norm}}}$$

Pour  $F \leq F_y$

$$\sigma_{\max} = \sigma_Y = 270 \text{ MPa}$$

$$\sigma_{\text{norm}} = \frac{\sigma_{\max}}{K} = \frac{270}{2.45} = 110 \text{ MPa}$$

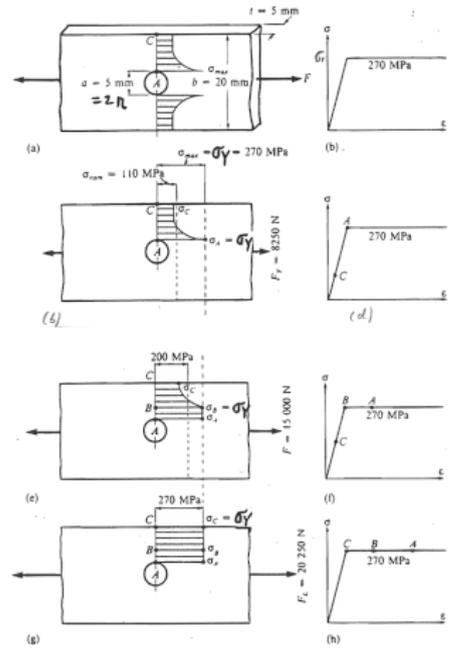
$$F_y = \sigma_{\text{norm}}(b - a) \cdot t = 110 \cdot (20 - 5)5 = 8250 \text{ N}$$

Pour  $F > F_y$

$$\sigma_{\text{norm}} = \frac{F}{A_{\text{norm}}} = \frac{15000 \text{ N}}{(20-5)5} = 200 \text{ MPa}$$

Pour  $F_L > F_y$

$$F_L = 270(20 - 5)5 = 20250 \text{ N}$$



## Mise en contexte – Fatigue

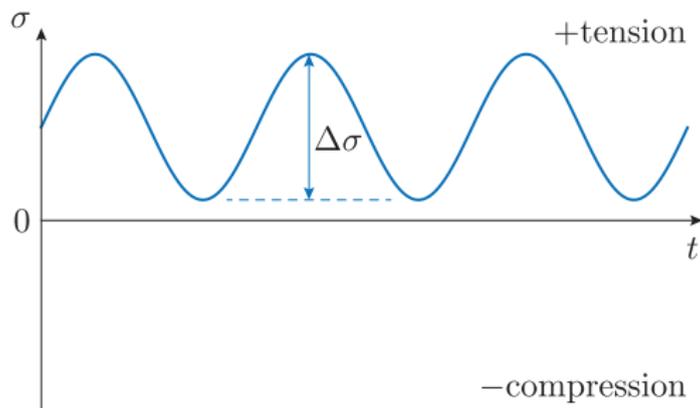


**Comment quantifier la résistance à la fatigue?**

[Google Images]

# Cycles de contraintes

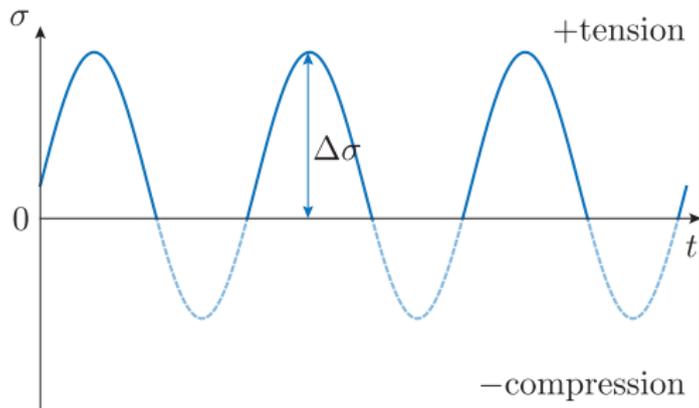
**2 facteurs** →  $\left\{ \begin{array}{l} N: \text{nombre de cycles} \\ \Delta\sigma: \text{amplitude des contraintes} \end{array} \right.$



**⚠ Fatigue → seulement pour la traction ⚠**

# Cycles de contraintes

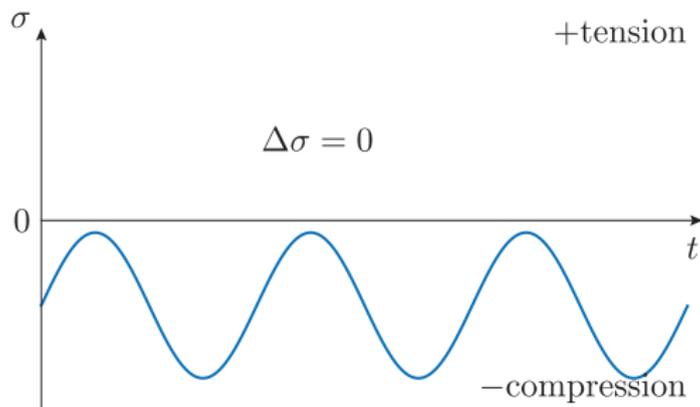
**2 facteurs** →  $\left\{ \begin{array}{l} N: \text{nombre de cycles} \\ \Delta\sigma: \text{amplitude des contraintes} \end{array} \right.$



**⚠ Fatigue → seulement pour la traction ⚠**

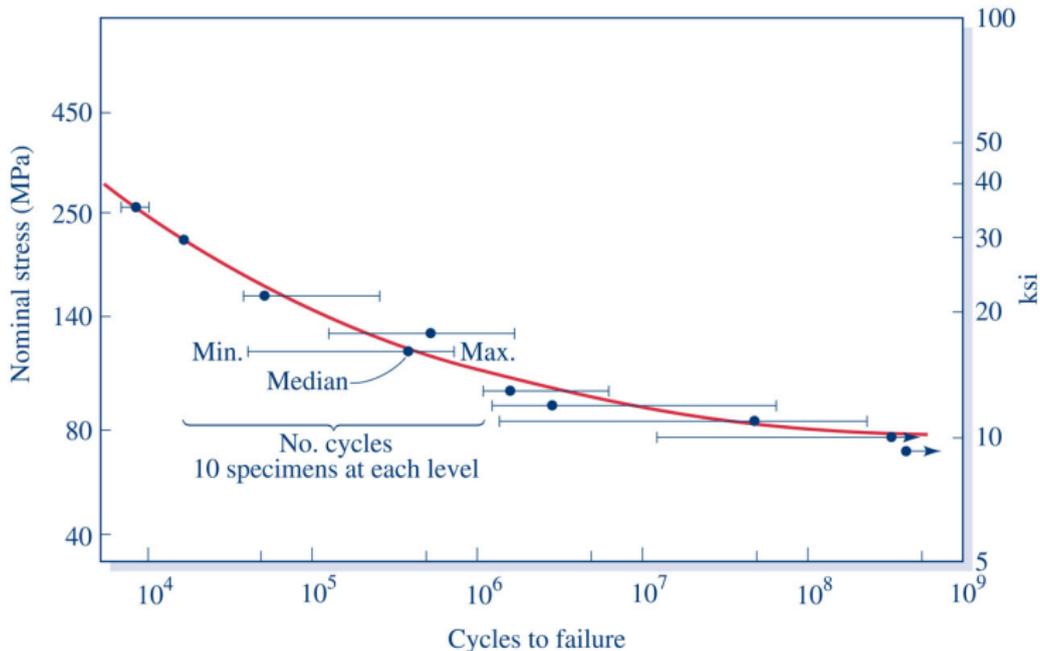
# Cycles de contraintes

**2 facteurs** →  $\left\{ \begin{array}{l} N: \text{nombre de cycles} \\ \Delta\sigma: \text{amplitude des contraintes} \end{array} \right.$



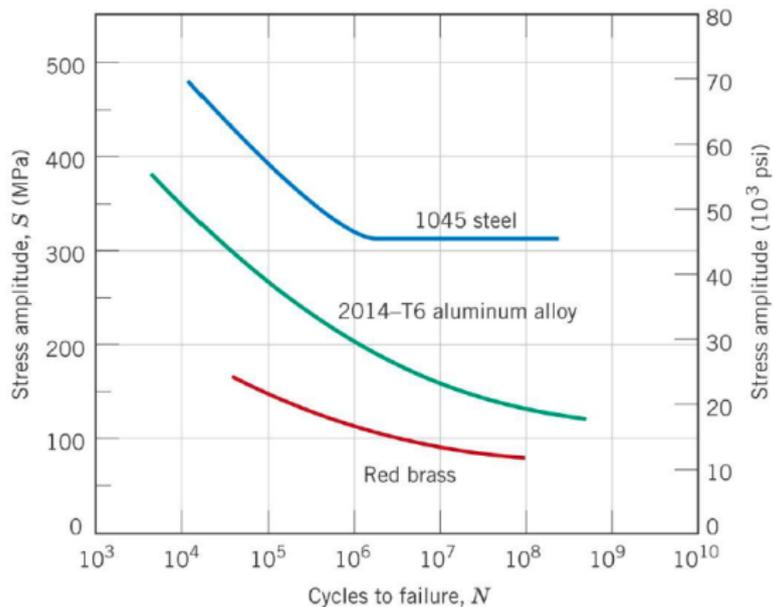
**⚠ Fatigue → seulement pour la traction ⚠**

# Courbes S-N



⚠ L'acier à une **limite d'endurance**

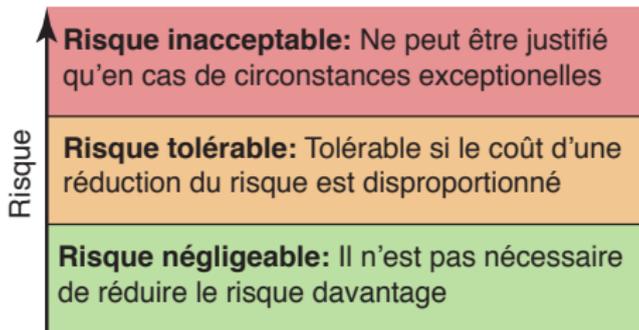
# Courbes S-N



⚠ L'acier à une **limite d'endurance**

# Risque

**Risque:** probabilité de défaillance  $\times$  conséquences



**Risque individuel:** La référence est habituellement la probabilité annuelle de décès ou blessure accidentelle

**Risque de société:** La référence est habituellement le nombre moyen de décès annuel attendu.



D. Diamantidis, *Background documents on risk assessment in engineering, risk acceptance criteria*, Tech. Rep. Document #3, JCSS - Joint Committee of Structural Safety, 2008

# Risque individuel acceptable, RI

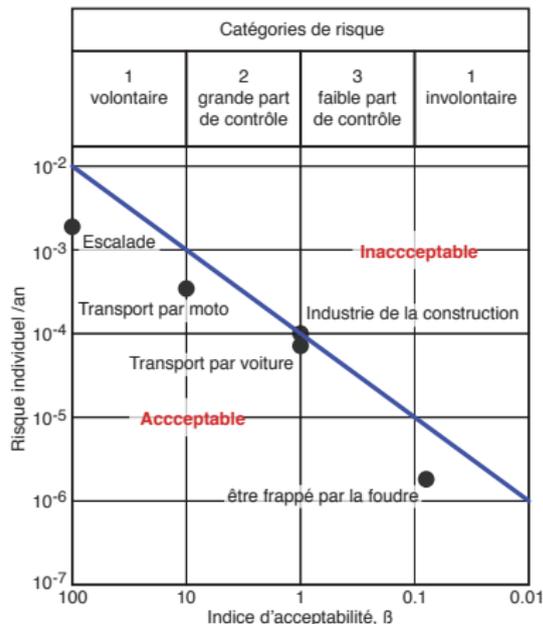
Exemple de limite de risque individuel acceptable

$$RI < \beta \times 10^{-4} / \text{an}$$



$$\approx \frac{26 \text{ semaines/an}}{14\,000\,000}$$

$$= 2 \times 10^{-6} / \text{an}$$



S. Jonkman, P. Van Gelder, and J. Vrijling, *An overview of quantitative risk measures for loss of life and economic damage*, Journal of Hazardous Materials, vol. 99, no. 1, pp. 1–30, 2003.

[Comité TAW, 1985]

# Risque de société acceptable

Exemple d'acceptabilité du risque pour des évènements touchant la société

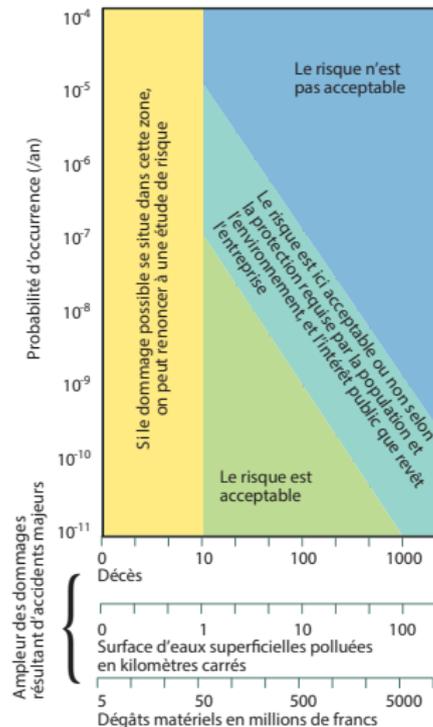


OFEFP, *Critères d'appréciation I pour l'ordonnance sur les accidents majeurs*, Office fédéral de l'environnement, des forêts et du paysage, Berne, 1996.

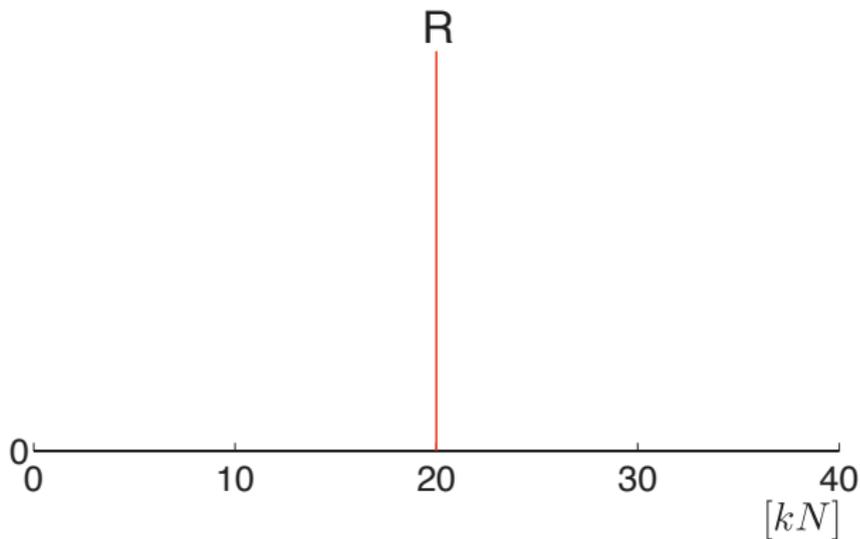
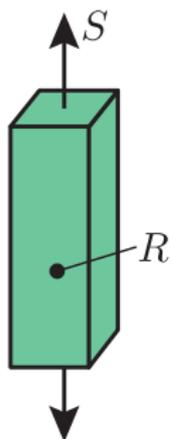


$$\approx \frac{26 \text{ semaines/an}}{14\,000\,000} = 2 \times 10^{-6} / \text{an}$$

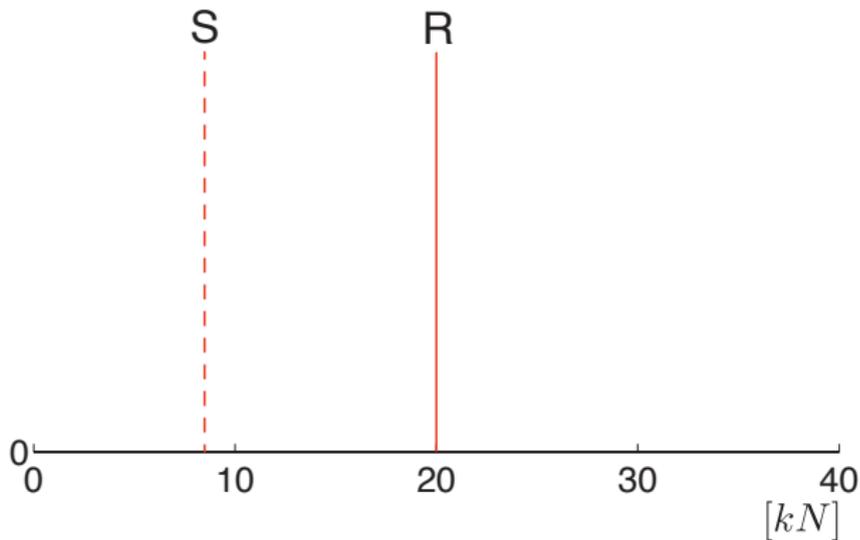
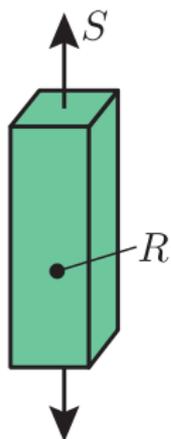
**Probabilité d'occurrence?**



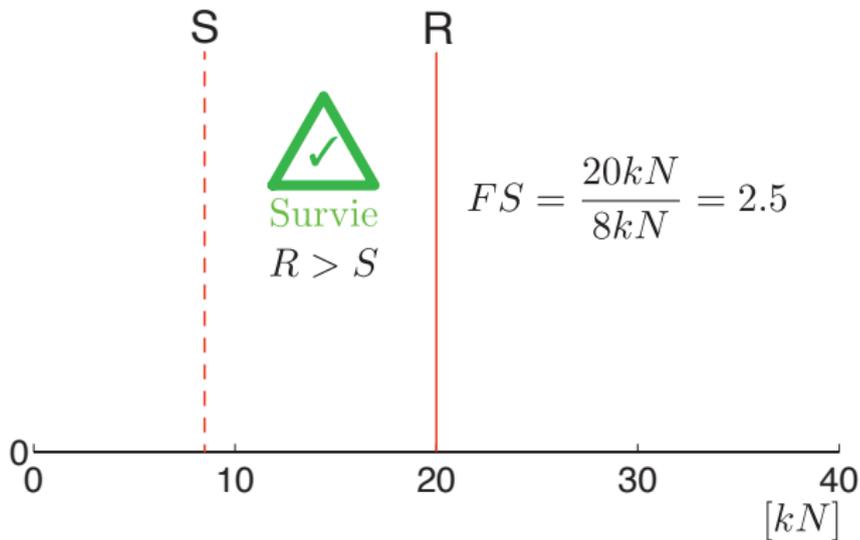
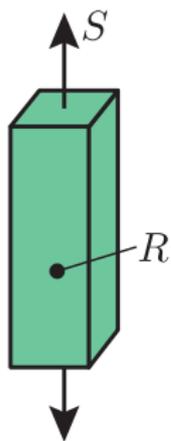
# Probabilité de défaillance



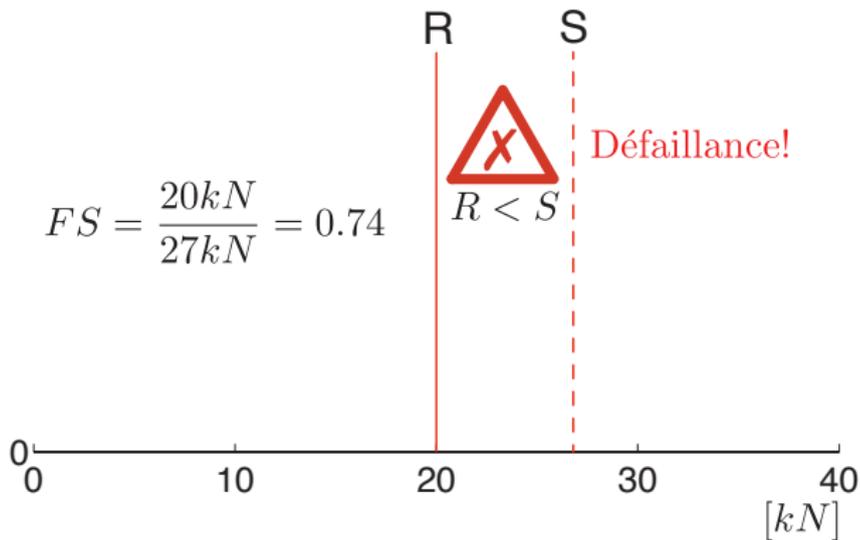
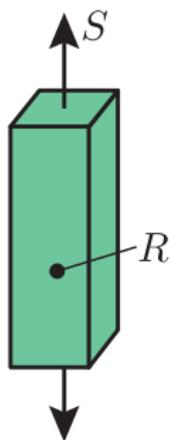
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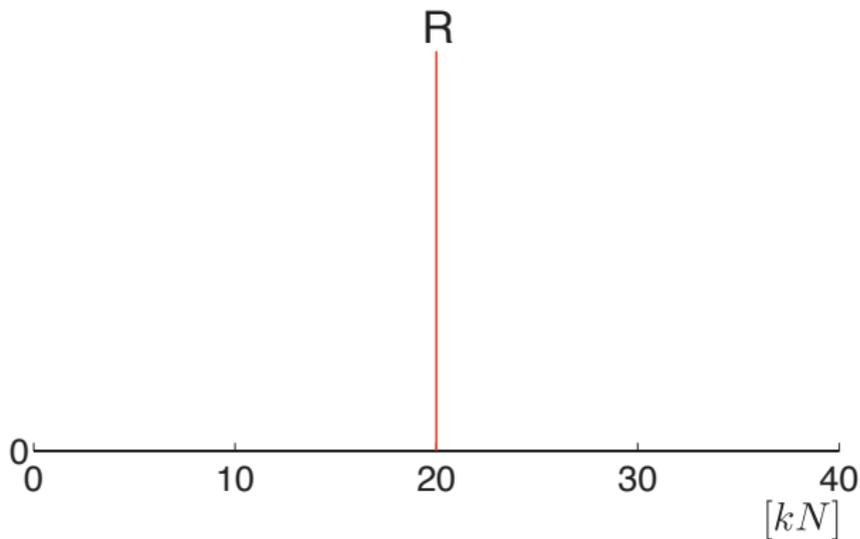
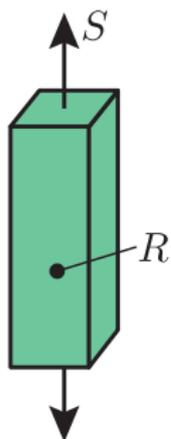
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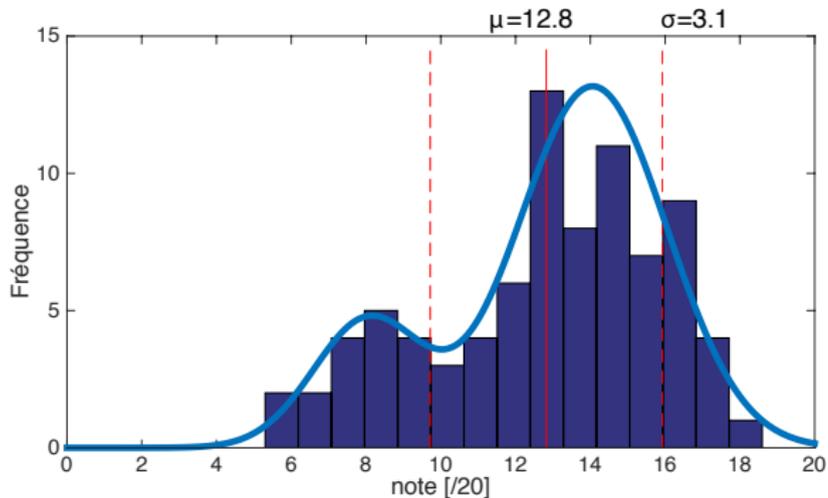
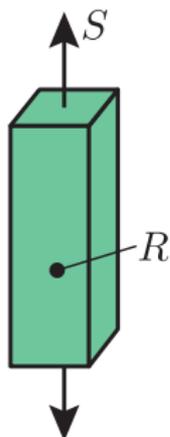
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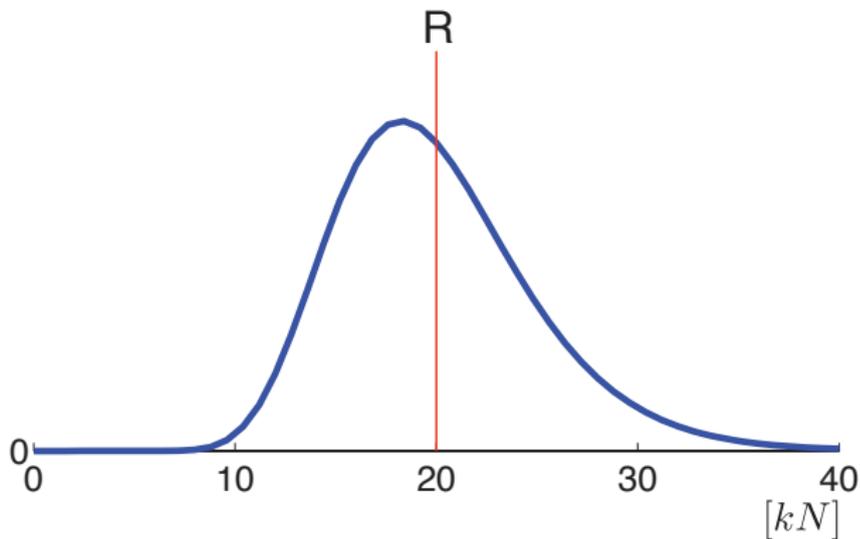
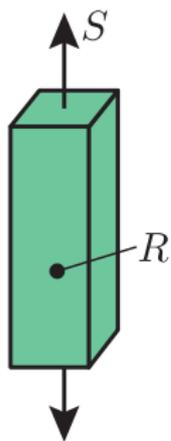
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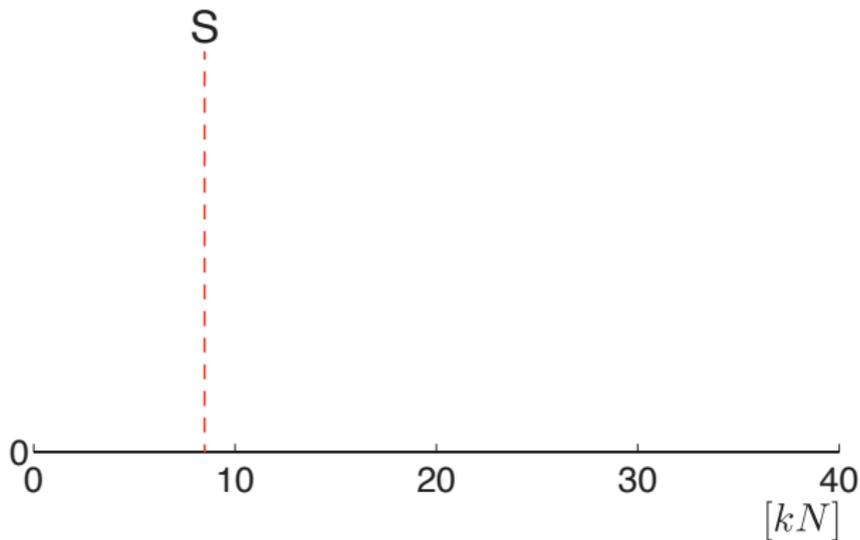
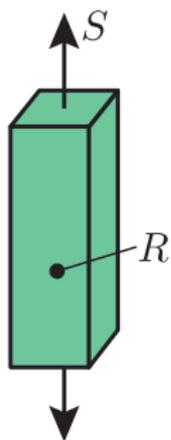
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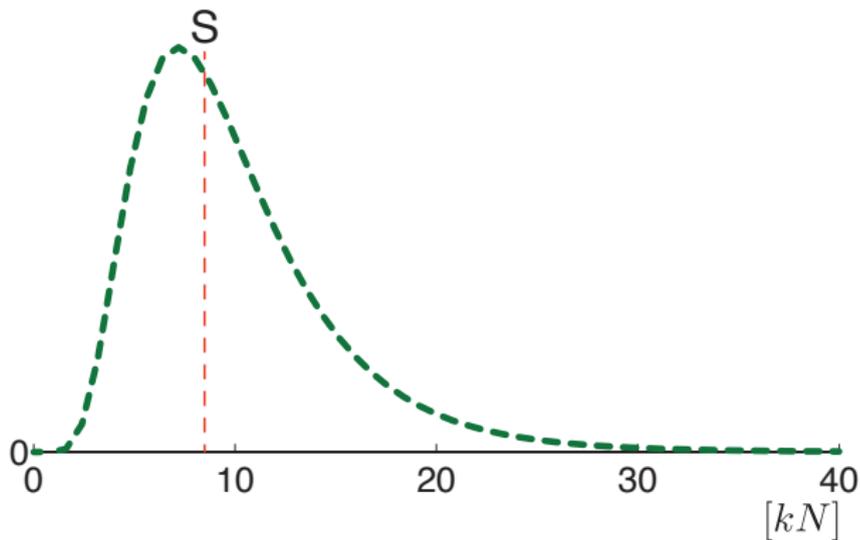
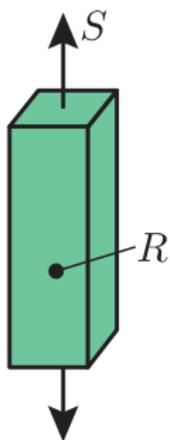
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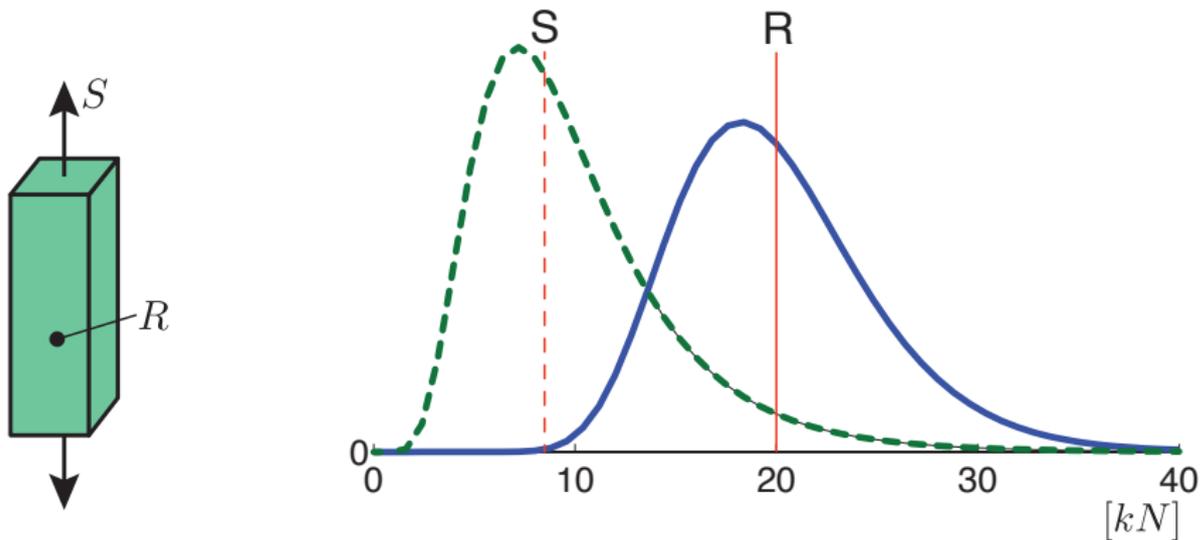
# Probabilité de défaillance



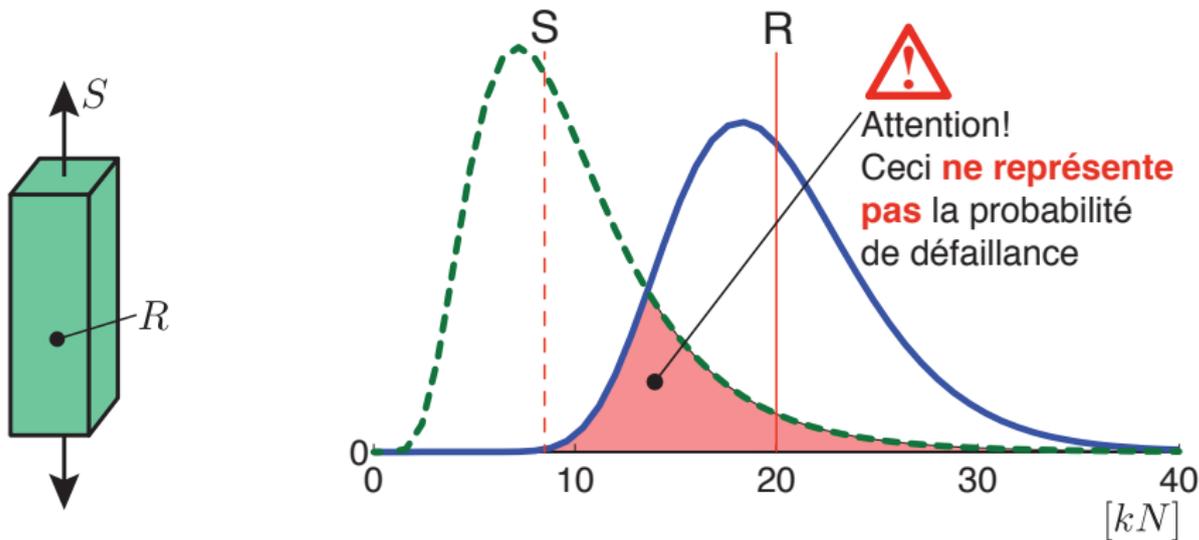
# Probabilité de défaillance



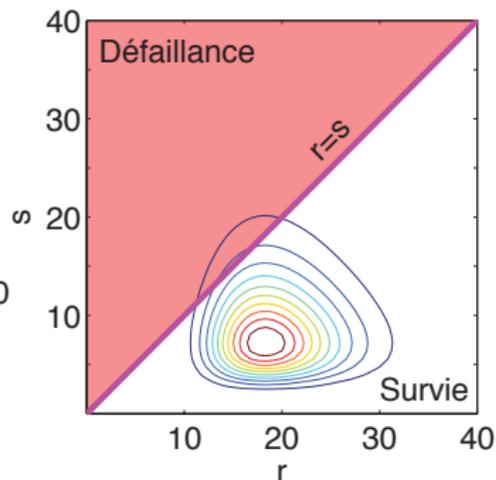
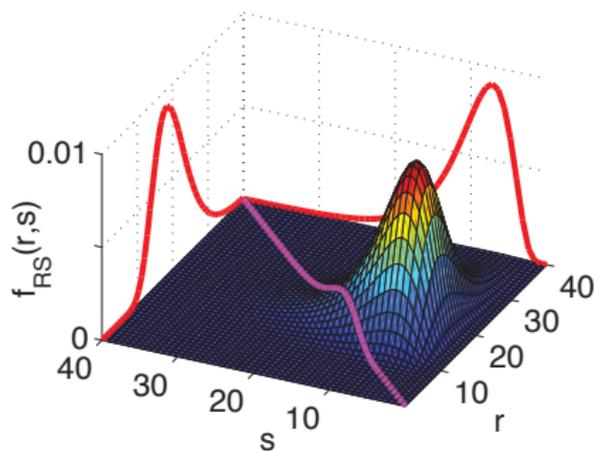
# Probabilité de défaillance



# Probabilité de défaillance



# Fiabilité des structures



**MTH2302C–Probabilités et statistique**  
**CIV8350–Fiabilité des structures et systèmes**

# Résumé – Module #10

## Critères de rupture

Matériau ductile: Von Mises & Tresca

Matériau fragile: Rankine & Mohr



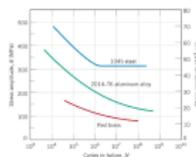
## Discontinuités géométriques

K: Facteur de concentration de contraintes

## Résistance à la fatigue

S: amplitude des cycles en traction

N: nombre de cycles

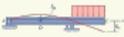
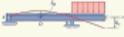


## Sécurité des ouvrages

$$\text{risque} = \text{conséquences} \times \text{probabilité}$$

[Popov]

## Organisation de la matière

- |                      |                 |   |  |  |
|----------------------|-----------------|---|--|--|
| 1                    | <i>Statique</i> | { | - Équilibre des forces et moments  |  |
|                      |                 |   | - Diagrammes de corps libres                    |  |
|                      |                 |   | - 5 Diagramme des efforts, $N(x)$ , $V(x)$ , $M(x)$  |  |
| 2                    | <i>Matériau</i> | { | - Contraintes & déformations   |  |
|                      |                 |   | - Loi de Hooke, Poisson & St-Venant             |  |
| <i>Chargements</i>   |                 | { | - 3 Efforts axiaux                              |  |
|                      |                 |   | - 4 Torsion                                     |  |
|                      |                 |   | - 6a Flexion                                    |  |
|                      |                 |   | - 6b Cisaillement                               |  |
|                      |                 |   | - 7 Déflexion                                   |  |
|                      |                 |   | - 9 Pression & chargements combinés            |  |
| <i>États limites</i> |                 | { | - 7 Déflexion                                   |  |
|                      |                 |   | - 8 Contraintes 2D-3D                           |  |
|                      |                 |   | - 10 Lois constitutives & critères de rupture  |  |